Perihelia reduction in the planetary problem

& applications

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The planetary problem

The Hamiltonian governing the motion of n planets and a star interacting through gravity only is:

$$H_{\rm planets,star}(u,v)=\sum_{i=0}^{n}\frac{|u_i|^2}{2\bar m_i}-\sum_{0\le i<j\le n}\frac{\bar m_i\bar m_j}{|v_i-v_j|}$$

$$\bar m_0=m_0:\ \ \text{sun}\ ,\quad \bar m_1=\mu m_1,\ \cdots,\ \bar m_n=\mu m_n:\ \ \text{planets}$$

$$u_i=(u_{i1},u_{i2},u_{i3})\in R^3\qquad v_i=(v_{i1},v_{i2},v_{i3})\in R^3\qquad v_i\neq v_j$$

$$\text{standard }2-\text{form}:\quad \Omega=\sum_{i=0}^{n}\sum_{j=1}^{3}du_{ij}\wedge dv_{ij}\qquad (3+3n\ \text{d. o. f.})$$

The heliocentric reduction of translation invariance

Translation invariance allows to study the motion of the relative positions

$$c_i x_i = v_i - v_0 \qquad i = 1, \cdots, n \ .$$

This is governed by the 3n--degrees of freedom Hamiltonian

$$\begin{aligned} H_{planets} &= \sum_{i=1}^{n} b_i \left(\frac{|y_i|^2}{2} - \frac{1}{|x_i|} \right) + \mu \sum_{1 \le i < j \le n} \left(\frac{1}{m_0} \frac{y_i \cdot y_j}{c_i c_j} - \frac{m_i m_j}{|c_i x_i - c_j x_j|} \right) \\ &= h_{int} + \mu f_{pert} \end{aligned}$$

where $\quad c_i, \ b_i = O_\mu(1) \qquad \Omega = \sum_{i=1}^{n} \sum_{j=1}^{3} dy_{ij} \wedge dx_{ij} \quad (3n \text{ d. o. f.})$

Symmetries

$H_{planets}$ is invariant:

a) by rotations:

$$(y_i, x_i) \to (R y_i, R x_i) \ \forall \ i \ , \qquad R \in SO(3) \ .$$

This is due to the three *independent, non-commuting* integrals

$$C = (C_1, C_2, C_3) = \sum_{i=1}^{n} x_i \times y_i \ ;$$

b) by reflections:

$$(y_i, x_i) \to (\sigma y_i, \tau x_i) \ \forall \ i \ , \qquad \sigma, \tau = \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix} \ .$$

The integrable part

Consider the integrable part

$$h_{int}(y_1, \cdots, y_n, x_1, \cdots, x_n) = \sum_{i=1}^{n} b_i \left(\frac{|y_i|^2}{2} - \frac{1}{|x_i|} \right)$$

assuming that

$$\frac{|y_i|^2}{2} - \frac{1}{|x_i|} < 0 \qquad \forall \ i = 1, \cdots, n \ .$$

Then the unperturbed motion for the coordinates

$$(y_i, x_i) \in R^3 \times R^3$$

evolves on Keplerian ellipses.

The ‘‘proper degeneracy’’

In action--angle coordinates,

$$\frac{|y_i|^2}{2} - \frac{1}{|x_i|} = -\frac{1}{2L_i^2} = -\frac{1}{2a_i}$$

is *one-dimensional* .

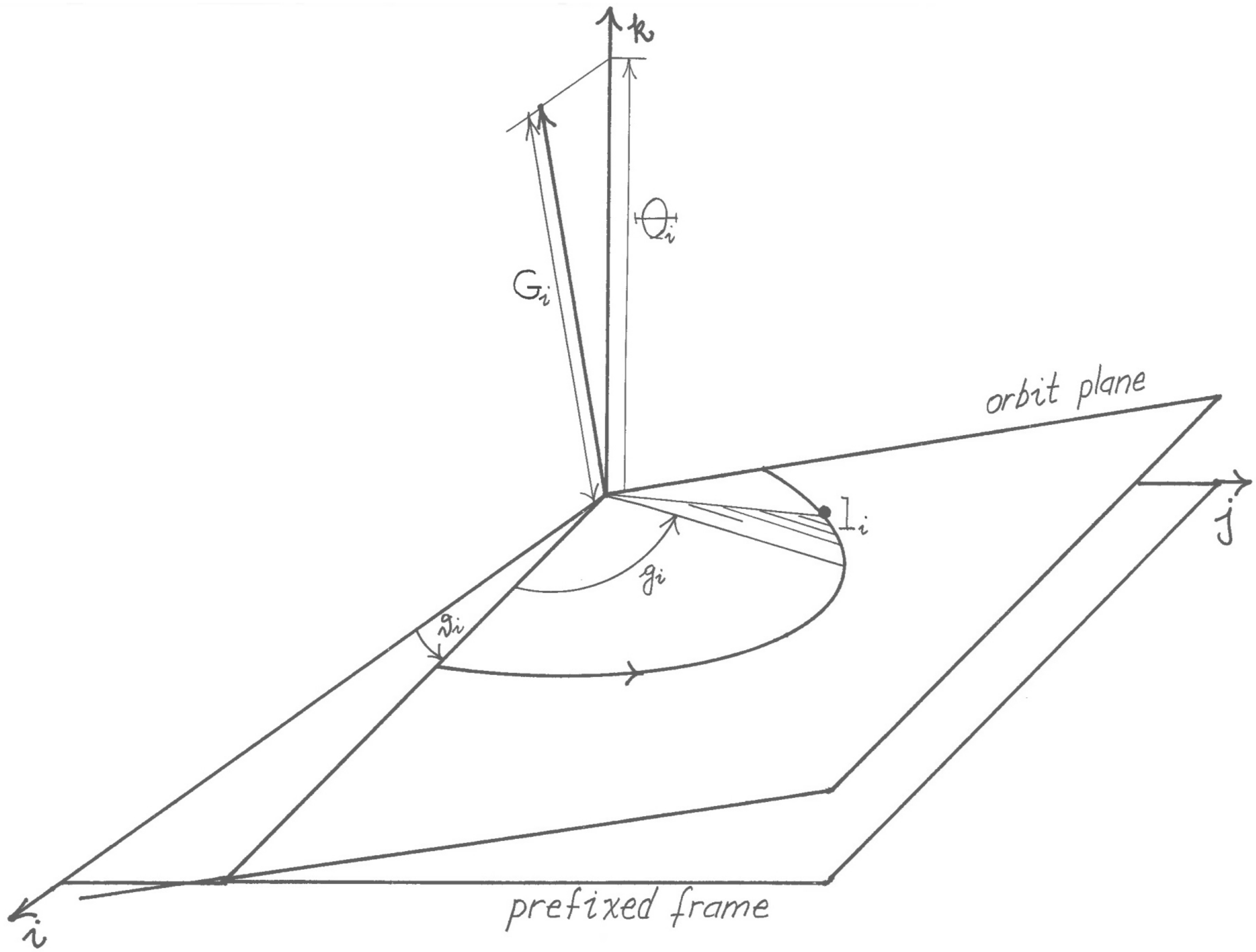
Therefore, the integrable part becomes

$$h_{Kep}(L_1, \cdots, L_n) = -\sum_{i=1}^{n} \frac{b_i}{2L_i^2}$$

a function of only on n action coordinates, out of 3n.

Canonical elliptic elements after Jacobi method

(‘‘Delaunay Coordinates’’)


k
Θ_i
G_i
orbit plane
l_i
j
g_i
ϑ_i
i
prefixed frame

$$\Omega_j = dL_i \wedge dl_i + dG_i \wedge dg_i + d\Theta_i \wedge d\vartheta_i = \sum_{j=1}^{3} dy_{ij} \wedge dx_{ij}$$

$$\Omega = \sum_{i=1}^{n} \Omega_j = \sum_{i=1}^{n} dL_i \wedge dl_i + dG_i \wedge dg_i + d\Theta_i \wedge d\vartheta_i$$

where: $x_i = (x_{i1}, x_{i2}, x_{i3}) \qquad y_i = (y_{i1}, y_{i2}, y_{i3})$

Poincaré Coordinates (H. Poincaré, 1892)

$$\mathrm{Poi} = (\Lambda, \ell, p, q) \in R^n \times T^n \times R^{2n} \times R^{2n}$$

$$p_j + i q_j = \sqrt{2(L_j - G_j)}e^{-i(g_j + \vartheta_j)} \qquad p_{j+n} + i q_{j+n} = \sqrt{2(G_j - \Theta_j)}e^{-i\vartheta_j}$$

$$(p_j \quad , q_j \quad) = (0, 0) \qquad \Longleftrightarrow \qquad \text{jth eccentricity} = 0$$

$$(p_{j+n}, q_{j+n}) = (0, 0) \qquad \Longleftrightarrow \qquad \text{jth inclination} = 0$$

$$\boxed{\Omega_j = dL_j \wedge d\ell_j + dp_j \wedge dq_j + dp_{j+n} \wedge dq_{j+n}}$$

The elliptic equilibrium (Arnold, 1963)

Consider the perturbing function expressed in Poincaré coordinates, averaged over ℓ_1, $\cdots$, ℓ_n:

$$\overline{f_{\rm Poinc}}(L, p, q) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} f_{\rm Poinc}(L, \ell, p, q) d\ell_1 \cdots d\ell_n \quad .$$

Then

$$(p, q) = 0$$

is an elliptic equilibrium point for $\overline{f_P}(L, p, q)$, for all $L = (L_1, \cdots, L_n)$.

Theorem on the stability of planetary motions (Arnold, 1963)

''For the majority of initial conditions under which the instantaneous orbits of the planets are close to circles lying in a single plane perturbations of the planets one another produce in the course of an infinite time little change of these orbits, provided the masses of the planets are sufficiently small.''

The long proof of Arnold's theorem

- V. I. Arnold, 1963: planar three--body problem. Uses Poincaré coordinates;

- J. Laskar & P. Robutel, 1995: spatial three--body problem. Use Jacobi reduction of the nodes;

- M. Herman & J. Féjoz, 2004: general problem. Use Poincaré coordinates + abstract arguments to reduce degeneracies (indirect proof);

- L. Chierchia & G. Pinzari, 2011: general problem. Use RPS coordinates (direct proof). Obtain sharp estimates on the measure of the invariant set.

All these proofs are based on KAM Theory.

a) Arnold's KAM Theory requires:

a1) $\overline{f}(L, p, q)$ is in BNF of sufficiently high order:

$$\overline{f}(L, p, q) = \sigma^0(L) + \sum_{j=1}^{m} \sigma_j^1(L)\tau_j + \frac{1}{2}\sum_{i,j=1}^{m} \sigma_{ij}^2(L)\tau_i\tau_j + \cdots \qquad \tau_i = \frac{p_i^2 + q_i^2}{2}$$

a2) $\det \sigma^2 \neq 0$ (twist condition).

Used by Arnold, 1963; Laskar & Robutel, 1994; Chierchia & P., 2011.

b) Herman's (weaker) KAM Theory requires that the frequency map

$$L \to \omega(L) := (\partial_L h_{Kep}(L), \sigma^1(L))$$

is non-planar:

$$\omega(L) \cdot c \neq 0 \qquad \text{forall } c = (c_1, \cdots, c_{n+m}) \in R^n \times R^m \ .$$

Used in an indirect way by Herman & Féjoz, 2004.

- Refined KAM theory is not enough to construct quasi-periodic motions.

- To construct BNF, in general, one needs

$$\sum_{i=1}^{m} \sigma_i^1 k_i \neq 0 \qquad \forall\ 0 \neq (k_1, \cdots, k_m) \in Z^m\ , \quad \sum_{i=1}^{m} |k_i| \leq M\ .$$

The degeneracy due to rotations, and Herman's

Consider the formal associated BNF

$$\sigma^0(\mathrm{L}) + \sum_{j=1}^{2n} \sigma^1_j(\mathrm{L})\tau_j + \frac{1}{2}\sum_{i,j=1}^{2n} \sigma^2_{ij}(\mathrm{L})\tau_i\tau_j + \cdots \qquad \tau_i = \frac{p_i^2 + q_i^2}{2}$$

Then

$$\sigma^1_{2n}(\mathrm{L}) \equiv 0 \qquad (\text{Arnold}, 1963)$$

$$\sum_{j=1}^{2n-1} \sigma^1_j(\mathrm{L}) \equiv 0 \qquad (\text{Herman}, 1990\text{s})$$

This prevents the construction of Birkhoff normal form, hence checking Arnold's condition, as well as checking Herman's condition.

The rotational degeneracy appears at any order of BNF

Moreover:

$$\sigma^{k}_{i_1,\cdots,i_k}(L) \equiv 0 \quad \text{if} \quad i_j = 2n \quad \text{for some } j \qquad (\text{Chierchia \& P. 2011}).$$

In particular:

$$\sigma^2_{i,2n} \equiv 0 \equiv \sigma^2_{2n,0} \quad \Rightarrow \quad \det \sigma^2(L) \equiv 0 \ .$$

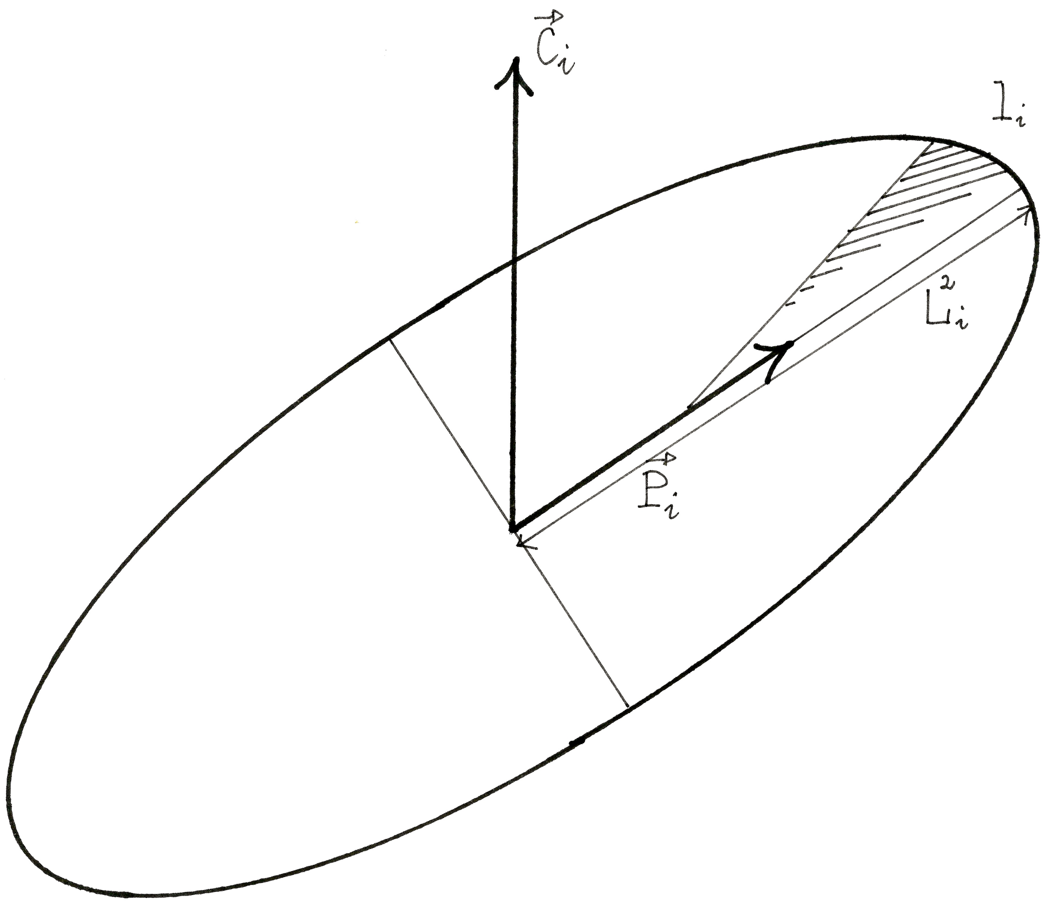
This violates Arnold's condition.

However...

The proper degeneracy allows for more freedom in the choice of coordinates.

Keplerian ellipses

$$\mathcal{E}_i = (L_i, l_i, C_i, P_i) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}^3 \times \mathbb{S}^2$$



L_i^2 = semimajor axis, l_i = mean nomaly

C_i = angular momentum, P_i = perihelion , $P_i \perp C_i$, $|P_i| = 1$

Good systems of coordinates

We like systems of coordinates of the form (L, l, Y, X) where

$$L = (L_1, \cdots, L_n) \ , \qquad l = (l_1, \cdots, l_n)$$

and the 4n coordinates

$$X = (X_1, \cdots, X_{2n}) \ , \qquad Y = (Y_1, \cdots, Y_{2n})$$

parametrize (possibly mixing) the planet's perihelia and angular momenta:

$$P_1, \cdots, P_n \ , \quad C_1, \cdots, C_n \ .$$

and, moreover

$$\Omega = \sum_{i=1}^{n} dL_i \wedge dl_i + \sum_{i=1}^{2n} dY_i \wedge dX_i \ .$$

Classical systems of coordinates (up to 2011)

1) Delaunay-Poincaré coordinates (∼1890; available for all n).

Widely used in the literature: Tisserand, Poincaré, Arnold, Nekhorossev, Robutel, Albouy, Herman, Féjoz, Kaloshin,...

2) Jacobi-Radau reduction of the nodes (1842; three bodies; reduce integrals, keep reflection symmetries)

Reduce the number of degrees of freedom by 2; bypass degeneracies. Used by Laskar & Robutel. Available just for n=2.

New entries (2009/2011--> present)

3) Boigey-Deprit coordinates (F. Boigey, 1982; A. Deprit, 1983);

Applications: unknown (out of the case n=2, where they reduce to Jacobi reduction of the nodes).

4) RPS coordinates (P. 2009, Chierchia & P. 2011).

Due to P. 2009, after rediscovering Deprit's coordinates. Reduce just one degree of freedom; bypass rotational degeneracy; keep symmetries (reflections and rotation around C). Allow for a direct proof of Arnold's theorem (Chierchia & P., Inv. Math., 2011).

5) Perihelia reduction (P. 2015).

Boigey-Deprit coordinates: main features

Boigey-Deprit coordinates are action-angle coordinates for n particles

$$(\mathrm{L}, \Gamma, \Psi, \mathrm{l}, \gamma, \psi) \in \mathrm{R}^{\mathrm{n}} \times \mathrm{R}^{\mathrm{n}} \times \mathrm{R}^{\mathrm{n}} \times \mathrm{T}^{\mathrm{n}} \times \mathrm{T}^{\mathrm{n}} \times \mathrm{T}^{\mathrm{n}}$$

Two couples of the

$$\Psi = (\Psi_1, \cdots, \Psi_{\mathrm{n}}) \qquad \psi = (\psi_1, \cdots, \psi_{\mathrm{n}})$$

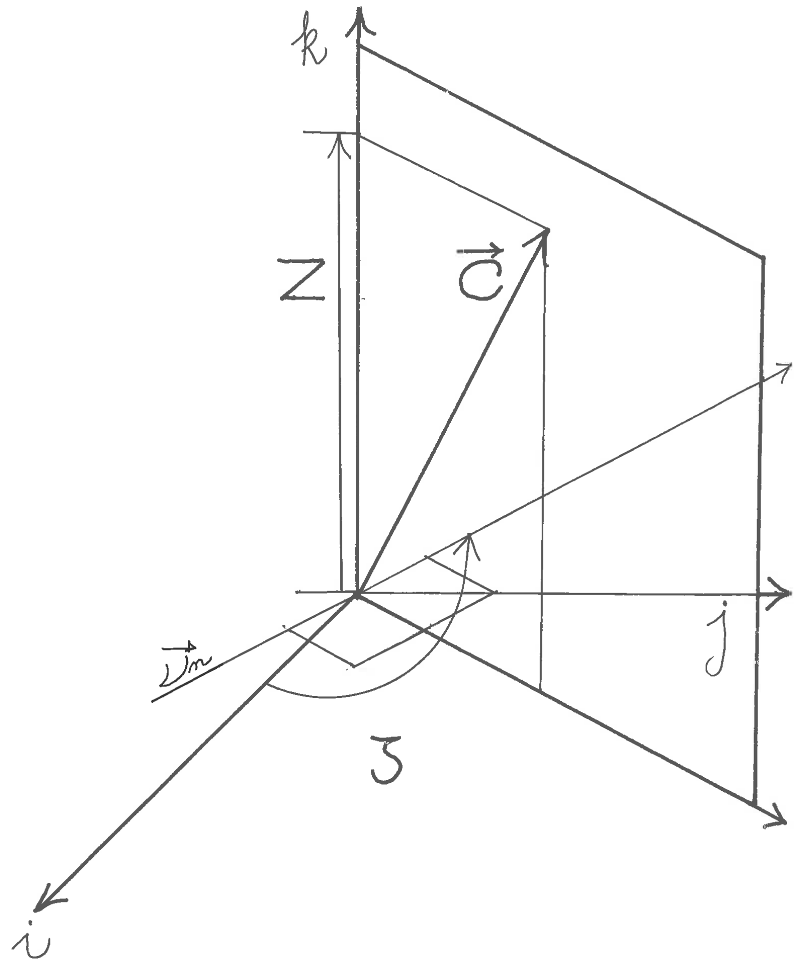
are mostly important:

$$\begin{cases} \Psi_{\mathrm{n}} := \mathrm{Z} \\ \psi_{\mathrm{n}} := \zeta \end{cases} \qquad \begin{cases} \Psi_{\mathrm{n}-1} := \mathrm{G} \\ \psi_{\mathrm{n}-1} := \mathrm{g} \end{cases}$$

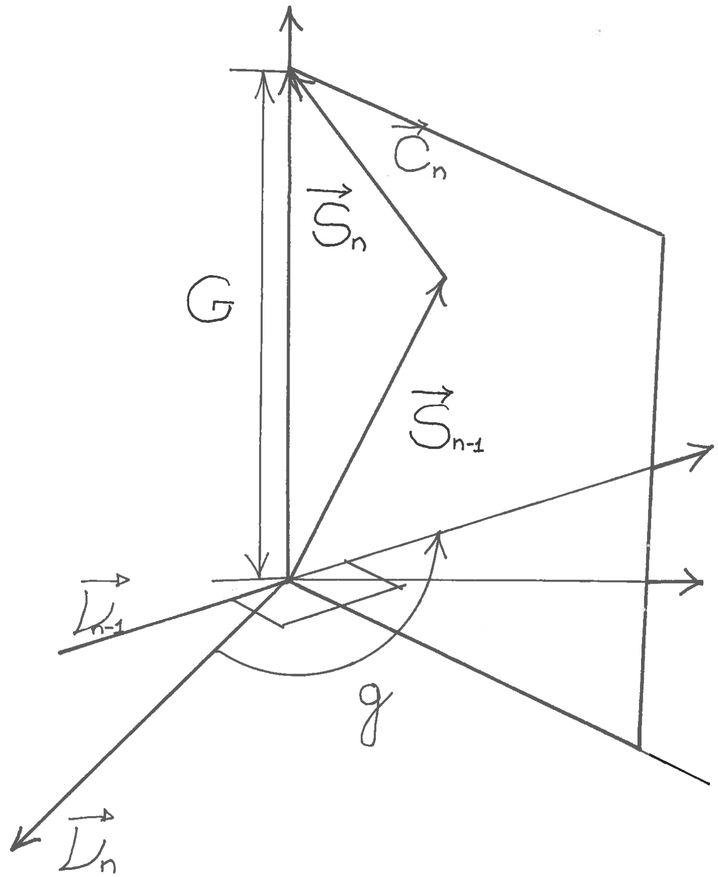
For a system of n particles in E=R^3, consider

$$C_j = x_j \times y_j \qquad S_j = \sum_{i=1}^{j} C_i \qquad S_n = C = (C_1, C_2, C_3): \quad \textit{integrals}$$

A negligible couple (i. e. the rotational degeneracy)

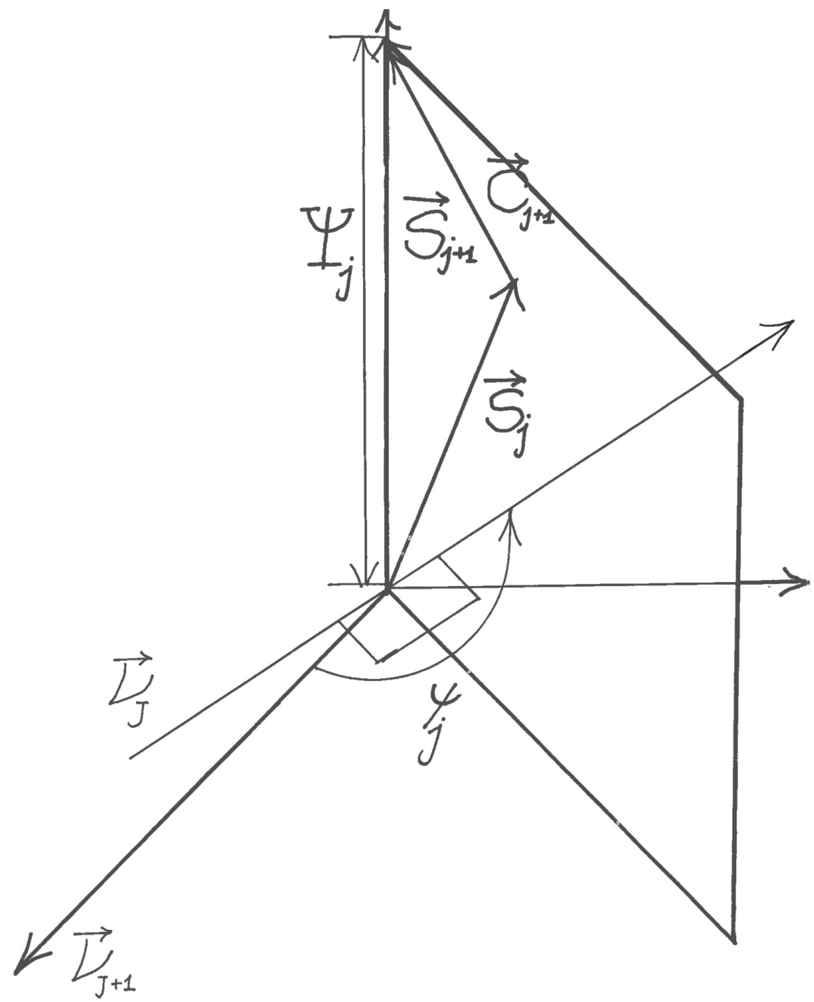


Generalizing Jacobi--Radau's construction



$\nu_{\mathrm{n}} = \mathtt{fixed}, \quad \nu_{\mathrm{n}-1} = \mathtt{line\ of\ the\ nodes,\ moving}$

Iteration



Regular Planetary Symplectic Coordinates

$$\begin{cases} \mathrm{L}_i \\ \ell_i = \mathrm{l}_i + \gamma_i + \sum_{i-1}^{n} \psi_k \end{cases} \qquad \psi_0 := 0$$

$$\begin{cases} \mathrm{p}_i = \sqrt{2(\mathrm{L}_i - \Gamma_i)}\cos\left(\gamma_i + \sum_{i-1}^{n} \psi_k\right) \\ \mathrm{q}_i = -\sqrt{2(\mathrm{L}_i - \Gamma_i)}\sin\left(\gamma_i + \sum_{i-1}^{n} \psi_k\right) \end{cases}$$

$$\begin{cases} \mathrm{p}_{j+n} = \sqrt{2(\Gamma_{j+1} + \Psi_{j-1} - \Psi_j)}\cos\sum_{i}^{n} \psi_k \\ \mathrm{q}_{j+n} = -\sqrt{2(\Gamma_{j+1} + \Psi_{j-1} - \Psi_j)}\sin\sum_{i}^{n} \psi_k \end{cases} \qquad 1 \leq j \leq n-1$$

$$\begin{cases} \mathrm{p}_{2n} = \sqrt{2(\mathrm{G} - \mathrm{Z})}\cos\zeta \\ \mathrm{q}_{2n} = -\sqrt{2(\mathrm{G} - \mathrm{Z})}\sin\zeta \end{cases}$$

(P. PhD, 2009).

Direct proof of Arnold's Theorem

- Let

$$\hat{p} = (p_1, \cdots, p_{2n-1}) \ , \qquad \hat{q} = (q_1, \cdots, q_{2n-1})) \ .$$

- The averaged perturbing function of the planetary problem

$$\overline{f_{RPS}}(L, \hat{p}, \hat{q}) = \frac{1}{(2\pi)^n} \int_{T^n} f_{pert}(L, \ell, \hat{p}, \hat{q}) d\ell_1 \cdots d\ell_n$$

can be put in BNF.

- This BNF has non trivial twist.

Three Remarks

- Jacobi--Radau, Boigey--Deprit's coordinates are not defined for the *planar* problem.

- Except for n=2, there are not parities associated to reflection symmetries. *In particular, the elliptic equilibrium is lost, under full reduction.*

- RPS coordinates are defined for the planar problem, but reduce just *one* d. o. f. Moreover, they are specific coordinates for the elliptic equilibrium.

Some questions concerning their application

- What about quasi-periodic motions *away* from the elliptic equilibrium?

- Do they exist?

- Which is their measure?

Theorem 1 [P. arXiv 2015, 78 p. A variant of Arnold's Theorem]

Let $n \in \mathbb{N}$, $n \geq 2$, $0 < e_{min} < e_{max} < 0.6627$. *Assume*

$$e_{min} \leq e_j \leq e_{max} \;, \quad a \leq a_* \;, \quad \frac{a_i}{a_{i+1}} \leq c a^{2^i} \;, \quad \text{and} \quad \mu \leq c a^{2^n} \;.$$

Then:

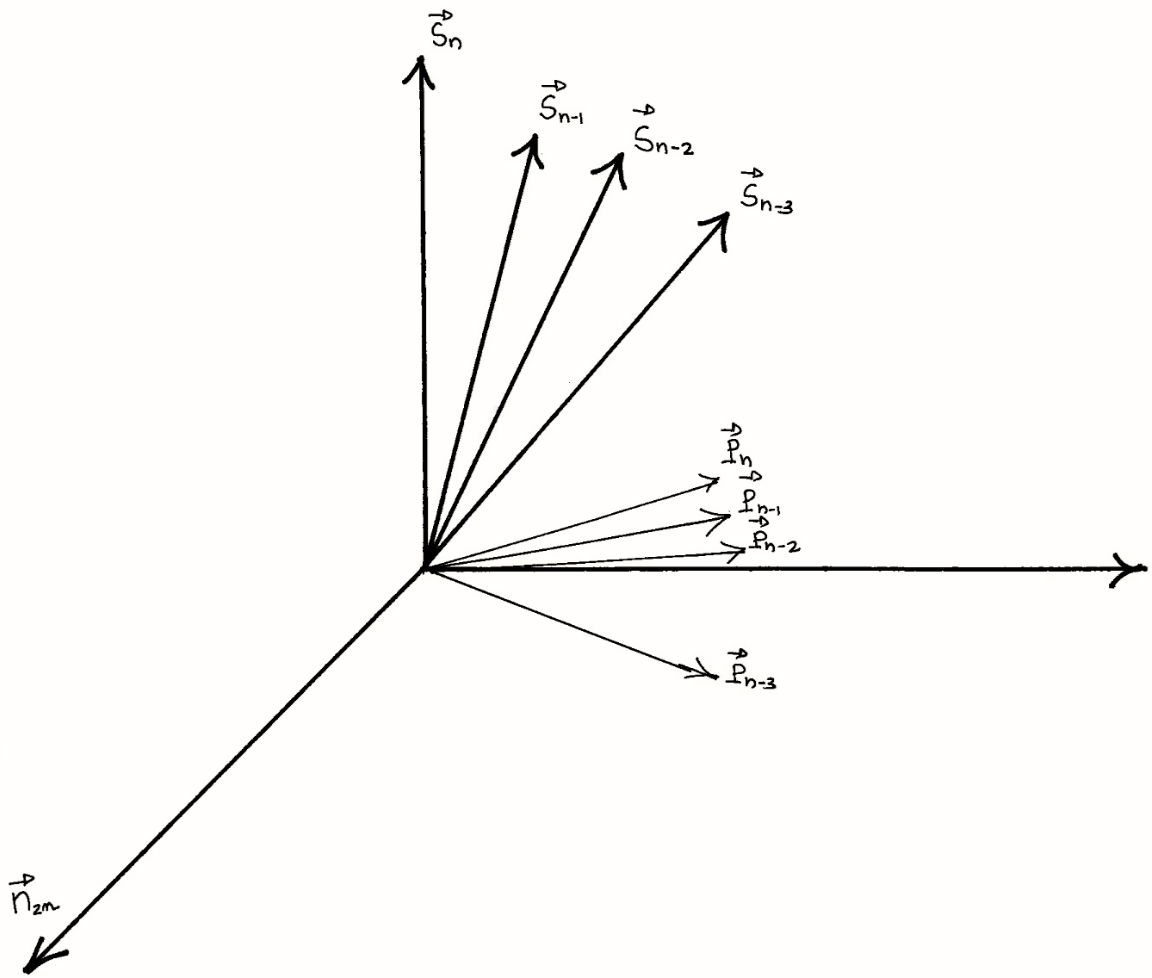
i) a positive measure ''Cantor'' set of (3n-2)-dimensional quasi-periodic motions exists, with

$$\frac{\mathrm{meas} K_{\mu,a}}{\mathrm{meas} B_a} \uparrow 1 \quad \text{as} \quad a \to 0$$

ii) Such set tends to the corresponding (2n-1)-dimensional set of motions of the planar problem when the inclinations tend to zero.

A related result has been announced by J. Féjoz since 2013.

The reduction of perihelia



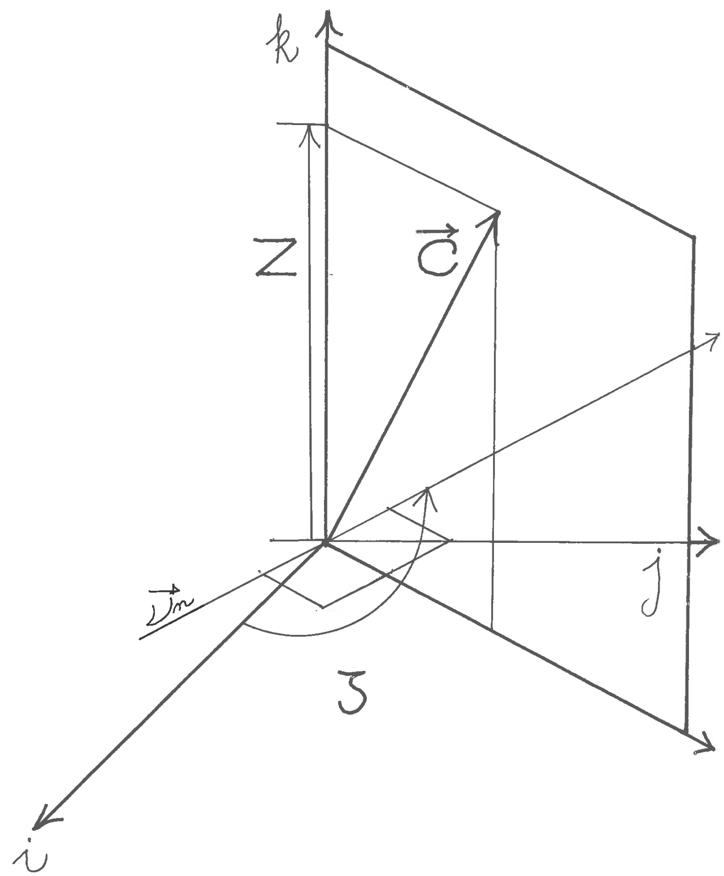
Consider the following sequence of ‘‘third axes’’

$$\begin{array}{ccccccccccccc} \mathtt{k} & \to & \mathtt{S_n} & \to & \mathtt{P_n} & \to & \mathtt{S_{n-1}} & \to & \cdots & \to & \mathtt{S_1} & \to & \mathtt{P_1} \\ \downarrow & & \| & & & & & & & & \| & & \\ \mathtt{arbitrary} & & \mathtt{C} & & & & & & & & \mathtt{C_1} & & \end{array}$$

where

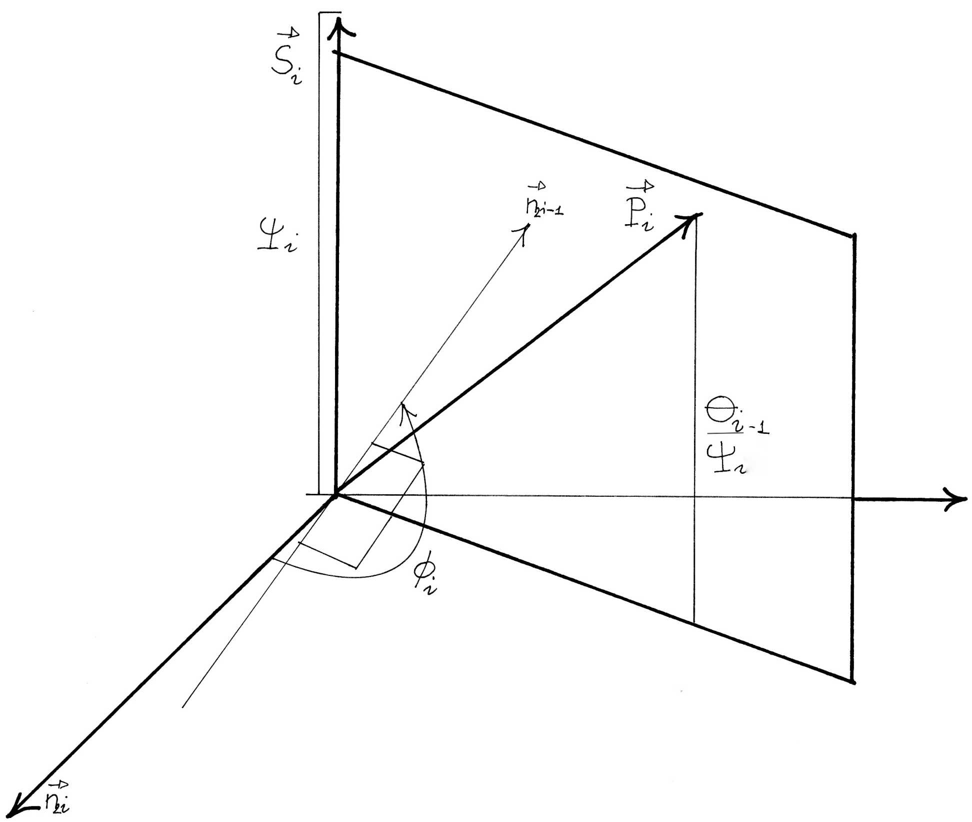
$$\mathtt{S_j} = \sum_{i=1}^{j} \mathtt{C_i} = \mathtt{jth\ partial\ angular\ momentum}, \qquad \mathtt{P_j} = \mathtt{jth\ perihelion}\ .$$

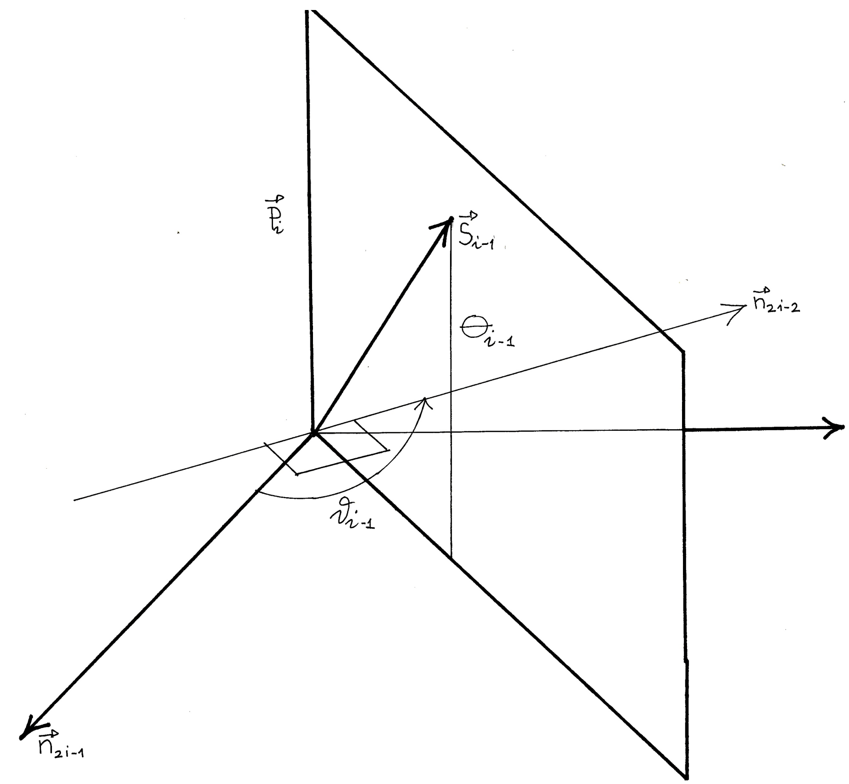
The coordinates $\mathcal{P} = (\mathtt{L}, \Psi, \Theta, \mathtt{l}, \psi, \theta)$



$$\Theta_n = Z, \qquad \theta_n = \zeta$$

The other coordinates





$\Psi_n = G \qquad \phi_n = \text{cyclic}$

$S_i \cdot P_i = S_{i-1} \cdot P_i := \Theta_{i-1}$

Novelties

- $\mathcal{P}$ coordinates are defined for *planar* problems. In fact, they are singular when $S_i \parallel P_i$, or $P_i \parallel S_{i-1}$.

- There is an *involution* associated to reflection symmetries:

$$\mathcal{R}: \ (\Theta, \Psi, L, \vartheta, \phi, l) \to (-\Theta, \Psi, L, -\vartheta, \phi, l) \quad \text{mod } 2\pi Z^n$$

- There are 2^{n-1} *planar equilibria*:

$$(\Theta_i, \vartheta_i) = (0, k_i \pi) \qquad k_i = 0, \ 1 \ , \qquad i = 1, \cdots, n-1$$

associated to $\mathcal{R}$.

- The tori of Theorem 1) are obtained as bifurcating from the equilibrium

$$S_1, \cdots, S_{n-1} = \uparrow\uparrow \cdots \uparrow \ .$$

The next challenge: to prove co-existence of stable and unstable motions

A result by Jefferys and Moser, 1966

In the three--body problem there exists a Cantor family of three--dimensional hyperbolic tori provided:

i) the planets' eccentricities are very small;

ii) their mutual inclination i is sufficiently large:

$$\cos^2 i < \frac{3}{5} \qquad (40^\circ < i < 140^\circ) \ .$$

NB: for small mutual inclination hyperbolic tori turn to be elliptic, with possibly different number of frequencies.
(Laskar & Robutel 1995; Biasco, Chierchia & Valdinoci, 2003).

New hyperbolic tori in the spatial 3BP (co-existence of stable and unstable motions)

Theorem 2 (P. work in progress)

In the three--body problem there exists a Cantor family of hyperbolic tori with three independent frequencies, provided:

i) *the mutual inclination is very small;*

ii) *the two planets revolve opposite one two the other, with the outer one clockwise with respect to C;*

iii) *the following conditions are satisfied*

$$\mathrm{L}_1 > \mathrm{L}_{1\mathrm{min}}(|\mathrm{C}|)\ ,\quad \mathrm{L}_2 > \mathrm{L}_{2\mathrm{min}}(|\mathrm{C}|)\ ,\quad |\mathrm{C}| < |\mathrm{C}_{\mathrm{outer}}| < |\mathrm{C}_{\mathrm{outer}}|_{\mathrm{max}}(\mathrm{L}_1, \mathrm{L}_2, |\mathrm{C}|).$$

Full Nekhorossev stability in spatial 3BP

Theorem 3 (P. work in progress)

All the motions of the three body problem where no mean motion resonances occur and the eccentricity of the inner planet is sufficiently small are Nekhorossev stable.

(Nekhorossev, 1977; Niederman, 1996): stability of semi--major axes; (P. 2013): full stability for planar 3BP.

The three--body secular Hamiltonian in Jacobi coordinates

$$\overline{f_{Jac}} = -\frac{a_1^2}{8a_2^3}\frac{L_2^3}{G_2^3} \times\Big((3\cos^2 i - 1)(5 - 3\frac{G_1^2}{L_1^2}) + 15\sin^2 i(1 - \frac{G_1^2}{L_1^2})\cos 2g_1\Big)$$

$$\cos i = \frac{G_1^2 + G_2^2 - G^2}{2G_1G_2}$$

(Harrington, 1969)

... and in $\mathcal{P}$-coordinates

$$\begin{aligned}\overline{f_{\mathcal{P}}} &= m_1 m_2 \frac{a_1^2}{4a_2^3}\frac{L_2^3}{G_2^5}\Big[\frac{5}{2}(3\Theta_1^2 - G_2^2) - \frac{3}{2}\frac{4\Theta_1^2 - G_2^2}{L_1^2}\Big(G^2 + G_2^2 - 2\Theta_1^2 \\ &\quad + 2\sqrt{(G_2^2 - \Theta_1^2)(G^2 - \Theta_1^2)}\cos\vartheta_1\Big) + \frac{1}{2}\frac{(G_2^2 - \Theta_1^2)(G^2 - \Theta_1^2)}{L_1^2}\sin^2\vartheta_1\Big] \\ &\quad + \mathrm{const} + \mathrm{h.o.t.}(a_1/a_2)\end{aligned}$$

Thank You!