On parameter loci of the Hénon family

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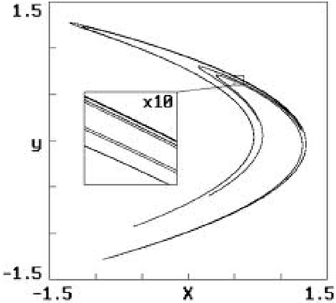
The Hénon family

Consider the *Hénon family* on $\mathbb{R}^2$:

$$f_{a,b} : (x, y) \longmapsto (x^2 - a - by, x), \quad (a, b) \in \mathbb{R} \times \mathbb{R}^\times.$$



We call $\mathbb{R} \times \mathbb{R}^\times$ the *parameter space* of the Hénon family $f_{a,b}$.

Two extremes

Fix $b \in \mathbb{R}^\times$ and change $a \in \mathbb{R}$ in $f_{a,b}(x, y) = (x^2 - a - by, x)$.

$a \ll 0 \Longrightarrow$ no periodic points of period ≤ 2.
$\Longrightarrow$ the non-wandering set $\Omega(f_{a,b})$ is empty.
$\Longrightarrow h_{\mathrm{top}}(f_{a,b}) = 0$.

$a \gg 0 \Longrightarrow f_{a,b}$ is a *hyperbolic horseshoe*, i.e.
$f_{a,b}|_{\Omega(f_{a,b})}$ is hyperbolic and conjugate to $\sigma : \{0, 1\}^{\mathbb{Z}} \circlearrowleft$.
$\Longrightarrow h_{\mathrm{top}}(f_{a,b}) = \log 2$ ($f_{a,b}$ attains the *maximal entropy*).

Theorem (Friedland-Milnor, 1989)

$0 \leq h_{\mathrm{top}}(f_{a,b}) \leq \log 2$ *holds for any* $(a, b) \in \mathbb{R} \times \mathbb{R}^\times$.

Main result 1

$$\mathcal{M}^\times \equiv \big\{(a,b) \in \mathbb{R} \times \mathbb{R}^\times : f_{a,b} \text{ attains the maximal entropy}\big\}$$

$$\mathcal{H}^\times \equiv \big\{(a,b) \in \mathbb{R} \times \mathbb{R}^\times : f_{a,b} \text{ is a hyperbolic horseshoe}\big\}.$$

Main Theorem (Arai-I, 2015)

$\exists$an analytic function $a_{\rm tgc} : \mathbb{R}^\times \to \mathbb{R}$ from the b-axis to the a-axis in the parameter space of $f_{a,b}$ with $\lim_{b\to 0} a_{\rm tgc}(b) = 2$ s.t.

- $a > a_{\rm tgc}(b)$ *iff* $(a,b) \in \mathcal{H}^\times$,
- $a \geq a_{\rm tgc}(b)$ *iff* $(a,b) \in \mathcal{M}^\times$.

Previously shown by Bedford and Smillie (2006) for $|b| < 0.06$.

Main result 2

Add the corresponding loci of the family $p_a(x) = x^2 - a$ $(a \in \mathbb{R})$:

$$\mathcal{M} \equiv \mathcal{M}^\times \cup \left\{(a, b) \in \mathbb{R}^2 : b = 0, a \geq 2\right\},$$

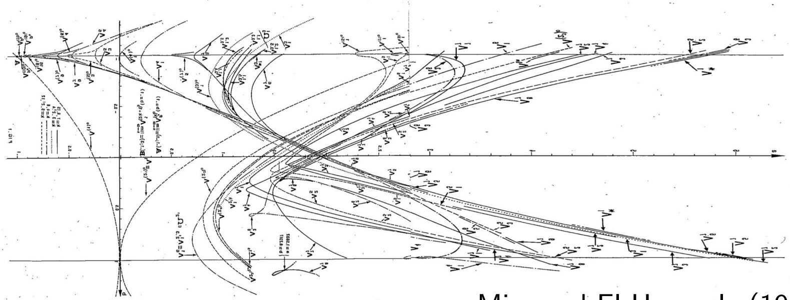
$$\mathcal{H} \equiv \mathcal{H}^\times \cup \left\{(a, b) \in \mathbb{R}^2 : b = 0, a > 2\right\}.$$

Corollary (Arai-I, 2015)

Both $\mathcal{H}$ and $\mathcal{M}$ are connected and simply connected subsets in $\mathbb{R}^2$. Moreover, we have $\overline{\mathcal{H}} = \mathcal{M}$ and $\partial\mathcal{H} = \partial\mathcal{M}$, and this boundary is piecewise real analytic with two analytic pieces.

According to Charles Tresser, our results were numerically observed about 35 years ago by Mira et al and Tresser as in the next figure.

Open questions



Mira and El Hamouly (1981)

Questions

(i) The function $a_{\rm tgc}$ *monotone on* $\{b > 0\}$ *and on* $\{b < 0\}$ *?*
(ii) Boundary of the zero-entropy locus piecewise analytic?
(iii) Complex horseshoe locus connected? (Not simply connected.)

Ingredients of proof

(i) Complex dynamics & complex analytic geometry:

- *Julia sets*,
- *crossed mapping condition (CMC)*,
- *Weierstrass preparation theorem*.

(ii) Rigorous numerics & smart numerical algorithms:

- *interval arithmetic*,
- *interval Krawczyk method*,
- *set-oriented computations*.

Quasi-trichotomy

We classify any Hénon map $f_{a,b}$ into three types (not exclusive).

Theorem (Quasi-Trichotomy)

$\exists$a piecewise affine function $a_{\mathrm{aprx}} : \mathbb{R}^\times \to \mathbb{R}$ so that

(1) $(a, b) \in \mathbb{R} \times \mathbb{R}^\times$ *with* $a \leq a_{\mathrm{aprx}}(b) - 0.1$

$\Longrightarrow h_{\mathrm{top}}(f_{a,b}) < \log 2$,

(2) $(a, b) \in \mathbb{R} \times \mathbb{R}^\times$ *with* $a \geq a_{\mathrm{aprx}}(b) + 0.1$

$\Longrightarrow f_{a,b}|_{\Omega(f_{a,b})}$ *is a hyperbolic horseshoe,*

(3) $(a, b) \in \mathbb{C} \times \mathbb{R}^\times$ *with* $|a - a_{\mathrm{aprx}}(b)| \leq 0.1$

$\Longrightarrow f_{a,b}$ *satisfies the CMC for projective boxes* $\{\mathcal{B}_i\}_i$ *in* $\mathbb{C}^2$.

Note: Indeed, (3) is valid for b in a complex neighborhood of $\mathbb{R}^\times$.
Let $\mathcal{F}$ be the set of $(a, b) \in \mathbb{C} \times \mathbb{C}^\times$ in the assumption of (3).

Non-maximal entropy

A complex Hénon map is $f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$, where $(a, b) \in \mathbb{C} \times \mathbb{C}^\times$.
A complex Hénon map $f_{a,b}$ is called *real* if $(a, b) \in \mathbb{R} \times \mathbb{R}^\times$.

(1) The proof relies on

Theorem (Bedford-Lyubich-Smillie (1993))

For a real Hénon map $f = f_{a,b}$, TFAE:

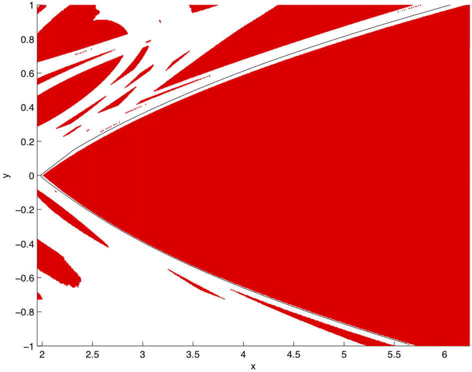
- $h_{\mathrm{top}}(f|_{\mathbb{R}^2}) = \log 2$,
- *the filled Julia set* $K_f \subset \mathbb{R}^2$,

where $K_f \equiv \left\{(x, y) \in \mathbb{C}^2 : \{f^n(x, y)\}_{n\in\mathbb{Z}} \text{ is bounded}\right\}$.

Our task is therefore to find a saddle periodic point in $\mathbb{C}^2 \setminus \mathbb{R}^2$.
This can be done by using interval Krawczyk method.

Hyperbolic horseshoes

(2) Apply algorithm of Arai (2007) to verify hyperbolicity on $\Omega(f)$.



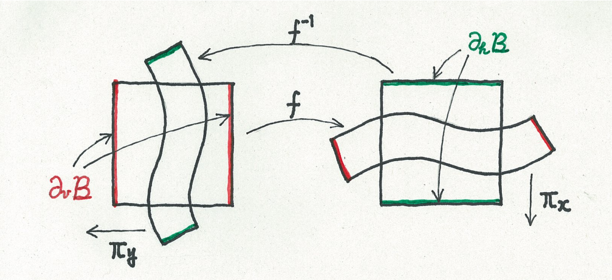
The red region is where uniform hyperblicity is rigorously verified.

Crossed mapping condition

A *box* is a product set $\mathcal{B} \equiv U_x \times U_y \subset \mathbb{C}^2$, where $U_x, U_y \subset \mathbb{C}$.
For a box $\mathcal{B} = U_x \times U_y$, set $\partial_v \mathcal{B} \equiv \partial U_x \times U_y$ and $\partial_h \mathcal{B} \equiv U_x \times \partial U_y$.

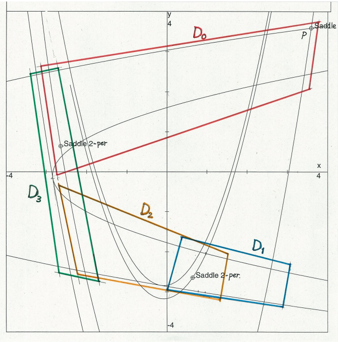
Definition

We say $f : \mathcal{B} \to \mathcal{B}$ satisfies the *crossed mapping condition (CMC)* if $\pi_x \circ f(\partial_v \mathcal{B}) \cap U_x = \emptyset$ and $\pi_y \circ f^{-1}(\partial_h \mathcal{B}) \cap U_y = \emptyset$ hold.



Verifying the CMC

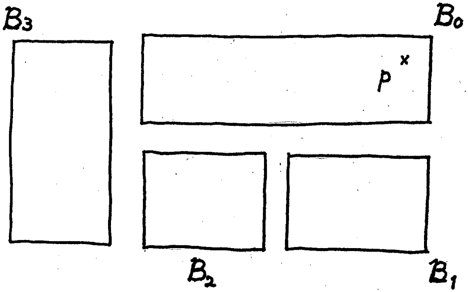
(3) Construction of projective boxes $\mathcal{B}_i$ associated with D_i



Draw a trellis, carefully choose quadrilaterals D_i which determine associated projective coordinates in $\mathbb{C}^2$, and construct boxes $\mathcal{B}_i$ with respect to the projective coordinates by fattening D_i into $\mathbb{C}^2$.

Symbolic decomposition

Family of boxes $\{\mathcal{B}_i\}_i$ enables us to define a symbolic encoding of the complex invariant manifolds $V^{u/s}(p)$ in $\mathbb{C}^2$, but with overlaps.



Here, p is the saddle fixed point in B_0.

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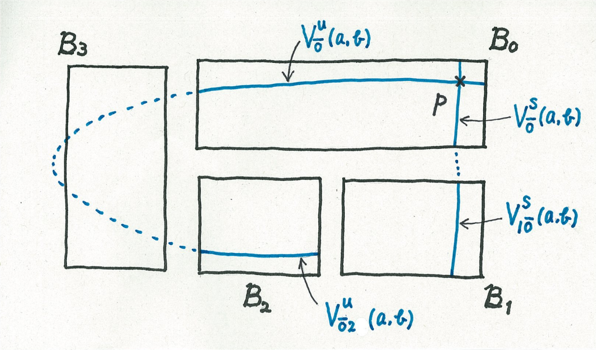

The CMC assures that these pieces of unstable/stable manifolds are horizontal/vertical holomorphic disks of some degree in $\mathcal{B}_i$.

Symbolic decomposition

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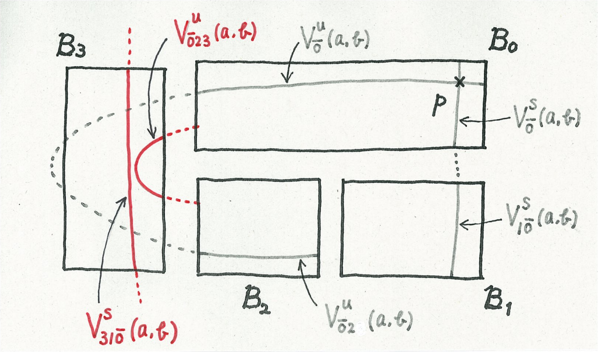

In the following discussion the two pieces $V^u_{\bar{0}23}(a,b)$ and $V^s_{31\bar{0}}(a,b)$ will play a cricial role.

Special pieces

Let $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ be real, $W_*^{u/s}(a,b)$ be real part of $V_*^{u/s}(a,b)$.
Plane topology argument + the CMC implies

Proposition (Special Pieces)

Suppose that $(a,b) \in \mathcal{F}_{\mathbb{R}} = \mathcal{F} \cap \mathbb{R}^2$.

1. *When* $b > 0$, $\mathrm{Card}(W^s_{31\bar{0}}(a,b) \cap W^u_{\bar{0}23}(a,b)) \geq 1$ *implies*
    - $W^s_{31\bar{0}}(a,b)$ *is the left-most among* $D_3 \cap W^s(p)$,
    - $W^u_{\bar{0}23}(a,b)$ *is the inner-most among* $D_3 \cap W^u(p)$.
2. *When* $b < 0$, $\mathrm{Card}(W^s_{41\bar{0}}(a,b) \cap W^u_{\overline{43}412}(a,b)) \geq 1$ *implies*
    - $W^s_{41\bar{0}}(a,b)$ *is the left-most among* $D_4 \cap W^s(p)$,
    - $W^u_{\overline{434}124}(a,b)_{\mathrm{inner}}$ *is the inner-most among* $D_4 \cap W^u(q)$.

Complex tangency loci

The previous proposition indicates that the two special pieces $W^s_{31\overline{0}}(a,b)$ and $W^u_{\overline{0}23}(a,b)$ (resp. $W^s_{41\overline{0}}(a,b)$ and $W^u_{\overline{4}3412}(a,b)$) are responsible for the last bifurcation for $b>0$ (resp. $b<0$).

Definition

We define

$$\mathcal{T}^+ \equiv \left\{(a,b) \in \mathcal{F} : V^s_{31\overline{0}}(a,b) \cap V^u_{\overline{0}23}(a,b) \neq \emptyset \text{ tangentially}\right\}$$

$$\mathcal{T}^- \equiv \left\{(a,b) \in \mathcal{F} : V^s_{41\overline{0}}(a,b) \cap V^u_{\overline{4}34124}(a,b) \neq \emptyset \text{ tangentially}\right\}$$

and call them the *complex tangency loci*.

Complex analytic sets

A *complex analytic set* is the set of common zeros of finitely many analytic functions in $U \subset \mathbb{C}^n$. A general consideration yields that $\mathcal{T}^{\pm}$ form complex analytic sets, but possibly with singularities.

How to wipe out the singularities?

Lemma

Let $U_a, U_b \subset \mathbb{C}$ and let $\mathcal{T} \subset U_a \times U_b$ be a complex analytic set. If $\overline{\mathcal{T}} \cap (\partial U_a \times U_b) = \emptyset$, the projection $\pi_b : \mathcal{T} \to U_b$ is proper.

Tansversality of $p_a(x) = x^2 - a$ at $a = 2 \Longrightarrow$ degree of π_b is one.
Proper of degree one $\Longrightarrow \mathcal{T}^{\pm}$ is complex manifold (no singularity!).

Proof of Main Theorem

Let $\mathcal{T}_{\mathbb{R}}^{+} \equiv \mathcal{T}^{+} \cap \mathbb{R}^2$.

Symmetry of $f_{a,b}$ w.r.t. complex conjugation $\Longrightarrow$

- $(a, b) \in \mathcal{T}_{\mathbb{R}}^{+}$ iff $W^s_{310\overline{0}}(a, b) \cap W^u_{\overline{0}23}(a, b) \neq \emptyset$ tangentially,
- the projection of $\mathcal{T}_{\mathbb{R}}^{+}$ to $\{b > 0\}$ is degree one,
- similar for $\mathcal{T}_{\mathbb{R}}^{-}$.

Therefore, we can define analytic $a_{\mathrm{tgc}} : \mathbb{R}^{\times} \to \mathbb{R}$ so that

$$(\text{the graph of } a_{\mathrm{tgc}}) = \mathcal{T}_{\mathbb{R}}^{+}$$

for $b > 0$ and similar for $b < 0$. Q.E.D.

Newton's method

Let $g : \mathbb{R}^n \to \mathbb{R}^n$. The Newton's method for solving $g(x) = 0$ is

$$N_g(x) = x - (Dg(x))^{-1} \cdot g(x).$$

For any invertible matrix C, the modified Newton's method is

$$N_{g,C}(x) = x - C \cdot g(x).$$

If $N_{g,C}(\Omega) \subset \mathrm{int}(\Omega)$ for $\Omega \subset \mathbb{R}^n$, then $\exists! x^* \in \Omega$ with $g(x^*) = 0$.

However, since

$$\mathrm{diam}(\Omega - C \cdot g(\Omega)) \approx \mathrm{diam}(\Omega) + \mathrm{diam}(C \cdot g(\Omega)) > \mathrm{diam}(\Omega),$$

it turns out that $N_{g,C}(\Omega) \subset \mathrm{int}(\Omega)$ <u>always</u> fails.

Interval Krawczyk method

Fix a base-point $x_0 \in \Omega$. By the (interval) mean-value theorem,

$$\begin{aligned} N_{g,C}(\Omega) &\subset N_{g,C}(x_0) + DN_{g,C}(\Omega) \cdot (\Omega - x_0) \\ &= x_0 - C \cdot g(x_0) + (\mathrm{Id} - C \cdot Dg(\Omega)) \cdot (\Omega - x_0). \end{aligned}$$

Definition

We call $K_{g,x_0,C}(\Omega) \equiv x_0 - C \cdot g(x_0) + (\mathrm{Id} - C \cdot Dg(\Omega)) \cdot (\Omega - x_0)$ the *interval Krawczyk operator*.

Choose C s.t. $C \cdot Dg(\Omega) \approx \mathrm{Id} \Longrightarrow \mathrm{diam}(K_{g,x_0,C}(\Omega)) < \mathrm{diam}(\Omega)$.

Theorem (Krawczyk, 1969)

(1) If $x^* \in \Omega$ *and* $g(x^*) = 0$*, then* $x^* \in K_{g,x_0,C}(\Omega)$.
(2) If $K_{g,x_0,C}(\Omega) \subset \mathrm{int}(\Omega)$*, then* $\exists! x^* \in \Omega$ *so that* $g(x^*) = 0$.

Set-oriented computations

Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $R \subset \mathbb{R}^n$ and take a decomposition $R = \bigcup_{i \in I} R_i$.
For $i \in I$, find a subset $I_i \subset I$ s.t.

$$f(R_i) \subset \bigcup_{j \in I_i} R_j.$$

Construct a directed graph $G = (V, E)$ as follows; we set $V = I$ and draw an arrow from $i \in I$ to $j \in I$ iff $j \in I_i$.
Then, we have

$$\mathrm{Inv}(f, R) \equiv \bigcap_{n \in \mathbb{Z}} f^n(R) \subset \bigcup_{i \in \mathrm{V}^{\pm\infty}(G)} R_i,$$

where $\mathrm{V}^{\pm\infty}(G)$ denotes the set of vertices $i \in V$ such that there exists a bi-infinite path through i.

Grazie!

Reference:
Z. Arai & Y. Ishii,
"On parameter loci of the Hénon family"
(arXiv:1501.01368)