On the fractal geometry of horseshoes in arbitrary dimensions

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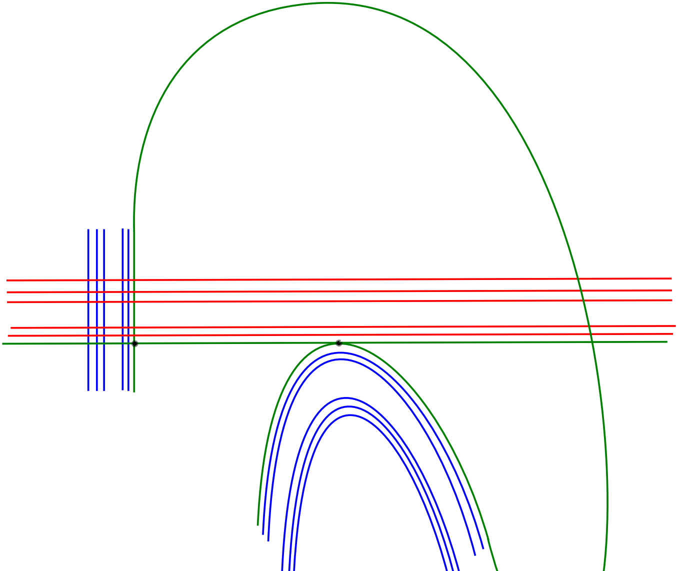
Original motivation: homoclinic bifurcations on surfaces



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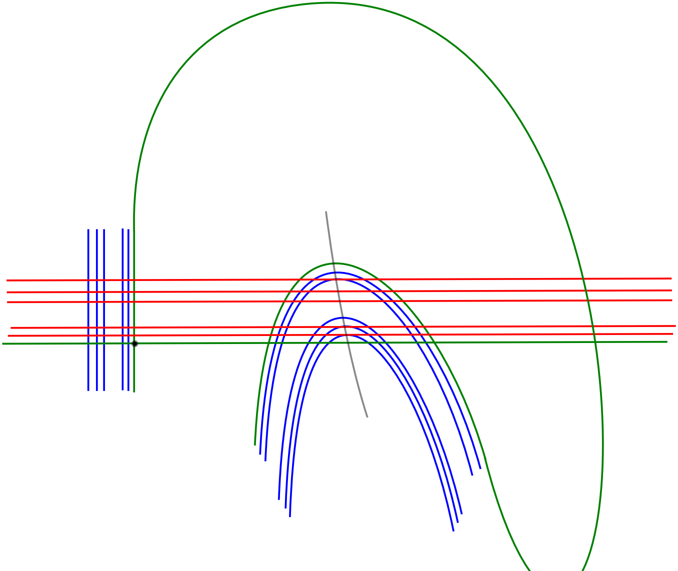
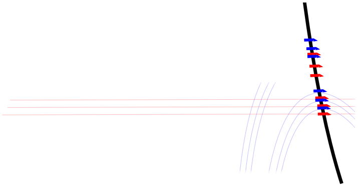


Remark

$HD(K^s) + HD(K^u) < 1 \Rightarrow HD(K^s - K^u) < 1$

M., Yoccoz, 2001

Typically, $HD(K^s) + HD(K^u) > 1 \Rightarrow K^s - K^u$ persistently contains intervals.



Remark

$HD(\Lambda) < 1 \Rightarrow HD(Proj(\Lambda)) < 1$

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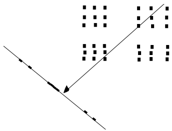
It follows from M., Yoccoz, 2001 that typically
$HD(\Lambda) > 1 \Rightarrow Proj(\Lambda))$ persistently contains intervals.



Palis, Viana, 1988

$HD(\Lambda)$ is continuous in the C^1-topology

Main techniques in M., Yoccoz, 2001:

- A *recurrent compact set criterion* for stable intersections (which implies that arithmetic differences persistently contain intervals).
- An application of Erdős probabilistic method: a family of C^∞ small perturbations of a regular Cantor set (the second Cantor set is fixed) with a large number of parameters such that for most parameters there is a recurrent compact set for the corresponding pair of Cantor sets.

Hausdorff dimension, HD

- $HD(\Lambda)$ is not always continuous (Bonatti, Díaz, Viana, 1995)
- Is $HD(\Lambda)$ generically continuous ?

Upper stable dimension, $\bar{d}_s$

- Homoclinic bifurcations in arbitrary dimensions (M., Palis, Viana, 2001)
- $\bar{d}_s(\Lambda) \geq HD(W^s_{loc}(\Lambda) \cap \Lambda)$

- Λ a horseshoe for a (local diffeomorphism f
- $\mathcal{P} = \{P_1, ..., P_m\}$ a Markov partition for Λ
- $\sigma : \Sigma \to \Sigma$ subshift of finite type conjugated to f^{-1} for $\mathcal{P}$
- $V_{\underline{\theta}} := \bigcap_{i=1}^{n} f^{-i}(P_{\theta_i})$, for $\underline{\theta} := (\theta_1, ..., \theta_n) \in \Sigma^*$
- $\Pi_s^{(j)}(V) := \sup_{x\in\Lambda\cap V}\{\prod_{i=1}^{j} \lambda_i(Df^n|_{E^s(x)})\}$, where $\lambda_i(A)$ denotes the i-th singular value of the linear map A.
- Σ^{+n}: words of size n starting at position 1

Definitions

- $d_n^{(1)}$ such that $\sum_{\underline{\theta}\in\Sigma^{+n}} \Pi_s^{(1)}(V)^{d_n^{(1)}} = 1$
- $\bar{d}_s^{(1)}(\Lambda) = \bar{d}_s(\Lambda) := \lim_{n\to\infty} d_n^{(1)}$

Definitions (cont.)

If, for a given j with $1 \leq j < k$, where k is the dimension of the stable spaces in Λ, $\bar{d}_s^{(j)}(\Lambda) > j$, define

- $d_n^{(j+1)}$ such that $\sum_{\underline{\theta} \in \Sigma^{+n}} \Pi_s^{(j)}(V)^{j+1-d_n^{(j+1)}} \Pi_s^{(j+1)}(V)^{d_n^{(j+1)}-j} = 1$
- $\bar{d}_s^{(j+1)}(\Lambda) := \lim_{n\to\infty} d_n^{(j+1)}$

If $\bar{d}_s^{(j)}(\Lambda) \leq j$, define $\bar{d}_s^{(r)}(\Lambda) := \bar{d}_s^{(j)}(\Lambda)$ for $j \leq r \leq k$. These definitions are inspired in the *affinity dimensions*, introduced by Falconer. We have analogous definitions for upper unstable dimensions.

Proposition

$\bar{d}_s^{(1)}(\Lambda) \geq \bar{d}_s^{(2)}(\Lambda) \geq \cdots \geq \bar{d}_s^{(k)}(\Lambda) \geq HD(\Lambda \cap W^s(x)), \forall x \in \Lambda$

Remark

If, for some $r \leq k$, $\overline{d}_s^{(r)}(\Lambda) < r$ then any image of any stable Cantor set of Λ by a C^1 (or Lipschitz) map on a manifold of dimension r has Hausdorff dimension smaller than r (and so zero Lebesgue measure).

Theorem (M., Palis, Viana)

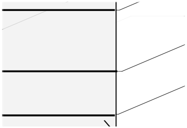
Given $r \leq k$ and $\delta > 0$ there is a $\delta-$small perturbation $\tilde{f}$ of f in the C^∞ topology and a subhorseshoe Λ' of the continuation of Λ for $\tilde{f}$ such that $\overline{d}_s^{(r)}(\Lambda') > \overline{d}_s^{(r)}(\Lambda) - \delta$ and Λ' has strong-stable foliations of codimension j for $1 \leq j \leq r$.

What follows is a joint work (in progress) with W. Silva.
From now on, we assume that Λ has strong-stable foliations of codimension j for $1 \leq j \leq r$.
We will introduce a concept of compact recurrent set, inspired by (M., Yoccoz, 2001) in order to obtain results like:

Proposition

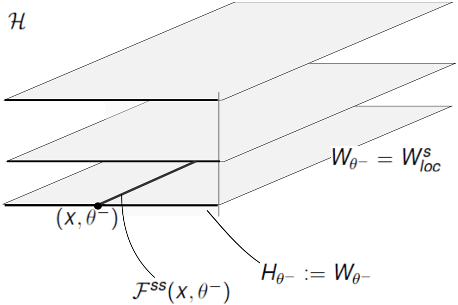
Assume that $\overline{d}_s^{(r)}(\Lambda) > r$. Then, perhaps after a small C^∞ perturbation, the images of stable Cantor sets by typical C^1 maps on r-dimensional manifolds persistently have non-empty interior.

Renormalization operators



- (f, Λ) horseshoe.
- $H := W^s_{loc}(\Lambda) \cap \mathcal{H}$, where $\mathcal{H}$ is some transversal to a codimension r strong-stable space E^{ss}.
- $H \approx I^r \times K^u$, where I is an interval on the line and K^u is a Cantor set.

Renormalization operators



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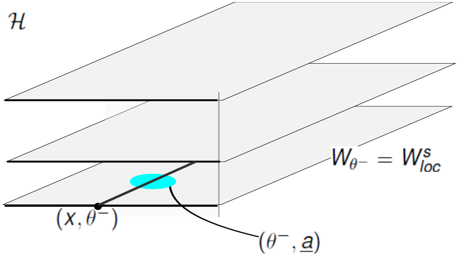

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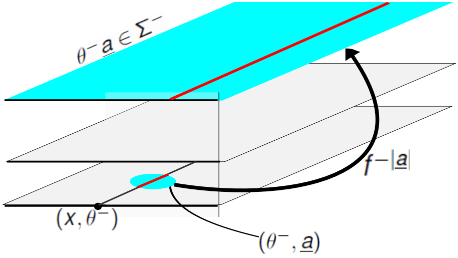

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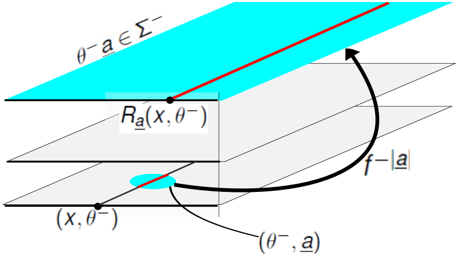

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Recurrent compact set

Definition

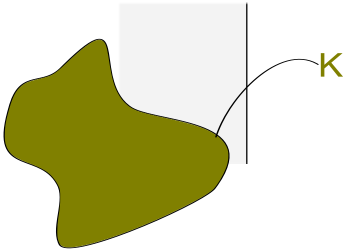
Let (f, Λ) a horseshoe.
$K \subset H$ is a **compact recurrent set** for (f, Λ) if:

- K is compact
- If $(\theta^-, x) \in K$, then there is a vertical cylinder corresponding to $\underline{a} \in \Sigma^{+*}$, such that $R_{\underline{a}}(\theta^-, x) \in int(K)$.

Remark

- We say that (f, Λ) satisfies the **compact recurrent criterium** (**CRC**) if there is a compact recurrent set for (f, Λ).
- The compact recurrent criterium is an open condition.

Robustness of the compact recurrent criterium



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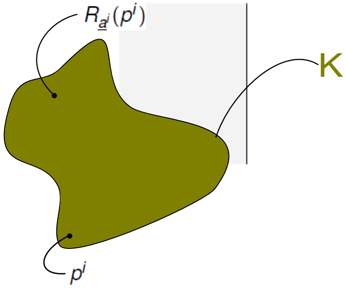

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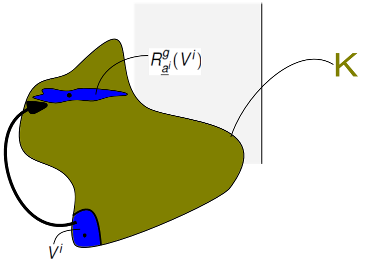
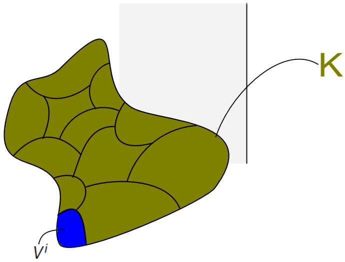


$R^g_{\underline{a}^i}(V^i) \subset int(K)$ for every $g \in B^{C^1}_{\delta_i}(f)$

Robustness of the compact recurrent criterium



- $K \subset \bigcup_{i=1}^{n} V^i$.
- $R^g_{\underline{a}^i}(V^i) \subset int(K)$ for every $1 \leq i \leq n$ and for every $g \in \bigcap_{i=1}^{n} B^{C^1}_{\delta_i}(f)$.

Blenders (Bonatti, Díaz)

Theorem 1

Se $(f, \Lambda) \in C^\infty$ satisfies the **CRC**, then it has a codimension r blender, C^1-persistently in a neighbourhood of $\mathcal{F}^{ss}_{loc}(K)$.

More specifically, any manifold sufficiently C^1-close to a leaf of the codimension r strong-stable foliation, $\mathcal{F}^{ss}_{loc}(x, \theta^-)$, through a point (x, θ^-) of K intersects $W^{g,u}(\Lambda^g)$ for any g sufficiently C^1-close to f.

Main Theorem

Let (f, Λ) be a C^∞ horseshoe. If $\overline{d}_s^{(r)}(\Lambda) > r$ then there is a horseshoe, (g, Λ^g), $C^\infty-$close to (f, Λ) which satisfies the **CRC.**

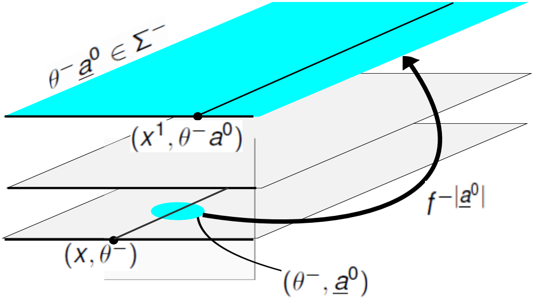
Corollary

Codimension r blenders appear, C^1-robustly, after a small C^k-perturbation of any C^k horseshoe (f, Λ) satisfying $\overline{d}_s^{(r)}(\Lambda) > r$.

Theorem 1' (restricted)

If (f, Λ) has a compact recurrent set K then $\mathcal{F}^{ss}_{loc}(x, \theta^-) \cap \Lambda \neq \emptyset$ for every $(x, \theta^-) \in K$. In other words, the projection along the leaves of the codimension r strong-stable foliation of the stable Cantor set $\Lambda \cap W^s_{\theta^-}$ contains $K \cap W^s_{\theta^-}$, and so has nonempty (r-dimensional) interior.

Proof of theorem 1' (simplified)



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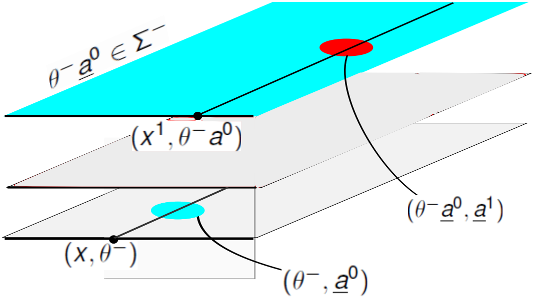

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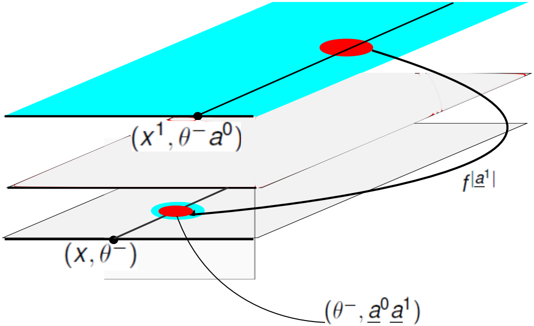

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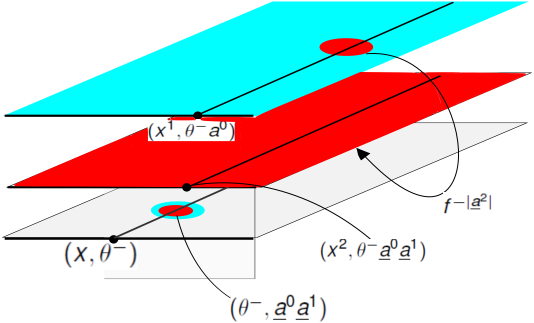

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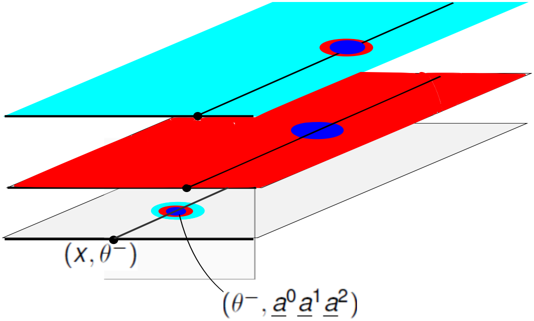

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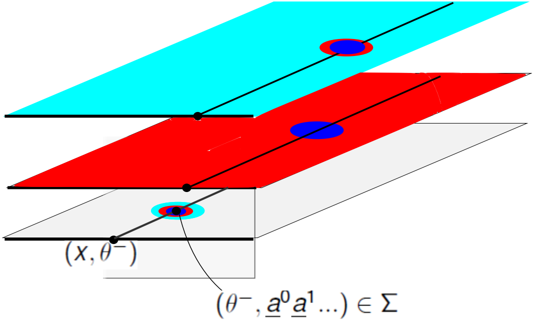
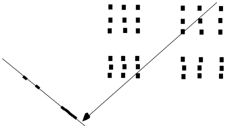


We first construct a good candidate for a compact recurrent set, and then we construct a family of small perturbations of the horseshoe with a large number of parameters and prove that for most parameters the candidate is indeed a recurrent compact set.
The main tool in the construction of the candidate for the recurrent compact set is inspired on the following classical result:

Marstrand, 1954

For Lebesgue almost every $\theta \in \mathbb{R}$, the projection of a set $K \subset \mathbb{R}^2$ with $HD(K) > 1$ along lines forming an angle θ with the horizontal axis x has positive Lebesgue measure.



We use the following generalization of Marstrand's theorem:

López, M., Silva

Let X be a compact metric space, $(\Lambda, \mathcal{P})$ a probability space and $\pi : \Lambda \times X \to \mathbb{R}^k$ a measurable function. Informally, one can think of $\pi_\lambda(\cdot) = \pi(\lambda, \cdot)$ as a family of projections parameterized by λ. We assume that for some positives real numbers α and C the following transversality property is satisfied:

$$\mathcal{P}[\lambda \in \Lambda : d(\pi_\lambda(x_1), \pi_\lambda(x_2)) \leq \delta d(x_1, x_2)^\alpha] \leq C\delta^k \qquad (1)$$

for all $\delta > 0$ and all $x_1, x_2 \in X$. Assume that $\dim X > \alpha k$. Then $Leb(\pi_\lambda(X)) > 0$ for a.e. $\lambda \in \Lambda$ and $\int_\Lambda Leb(\pi_\lambda(X))^{-1} d\mathcal{P} < +\infty$.