Rempe model and Denjoy odometer for hedgehogs of complex quadratic polynomials

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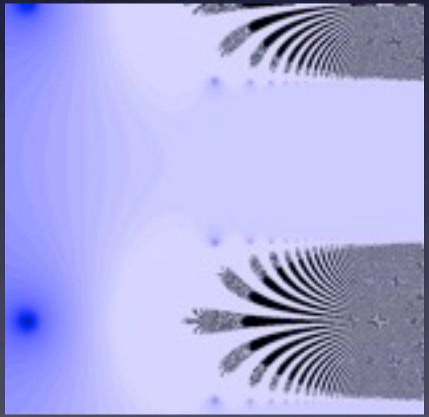
$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots$ holomorphic or polynomial

$|\lambda| = e^{2\pi i \alpha}, \ \alpha \in \mathbb{R} \setminus \mathbb{Q}$ irrationally indifferent fixed pt

Coexistence of different dynamical behaviors

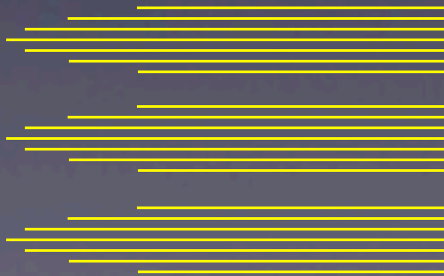
- Rotation-like
  - conjugate to a rotation
  - NOT conjugate to a rotation but still moves under the influence of the rotation
    - hedgehogs defined by Perez Marco
- Chaotic

Similarity to the Julia set of exponential maps



Straight brush model by Rempe

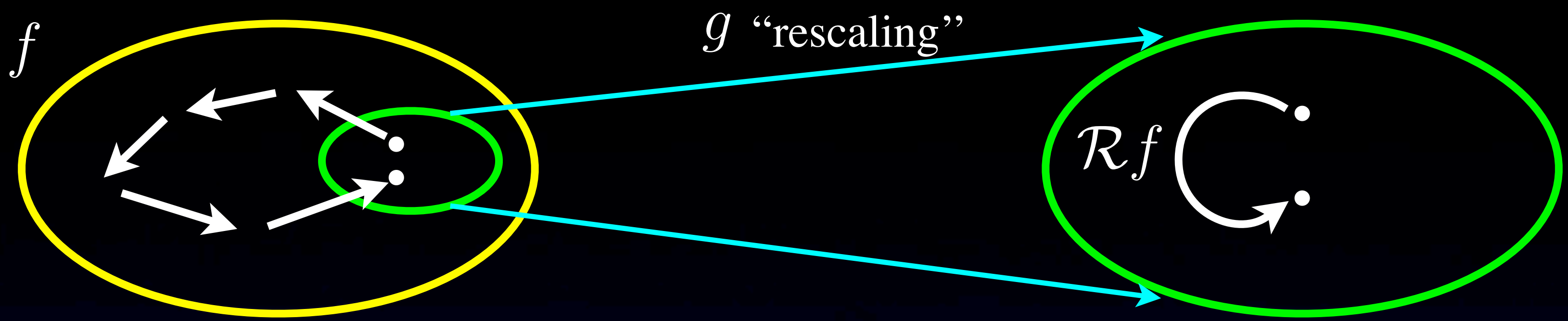
Renormalization for irrat. indiff. fixed pts



Near-parabolic (cylinder) renormalization

Dynamical charts and Denjoy odometer

Return map and renormalization

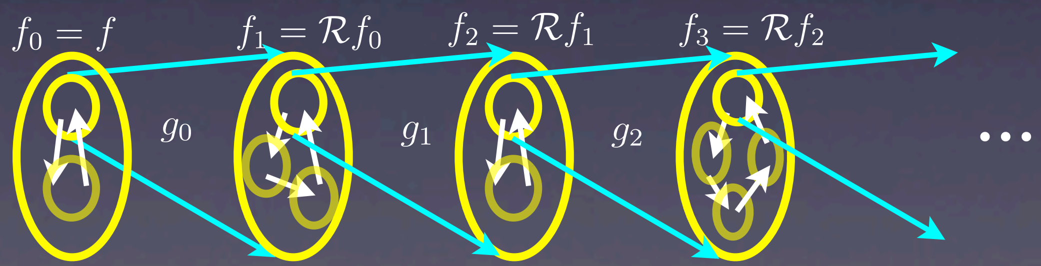


$$\mathcal{R}f = (\text{first return map of } f) \text{ after rescaling}$$
$$= g \circ f^k \circ g^{-1} \quad (\text{if return time} \equiv k)$$

Renormalization

high iterates of f	$\longleftrightarrow$	fewer iterates of $\mathcal{R}f$
fine orbit structure for f	$\longleftrightarrow$	large scale orbit structure for $\mathcal{R}f$

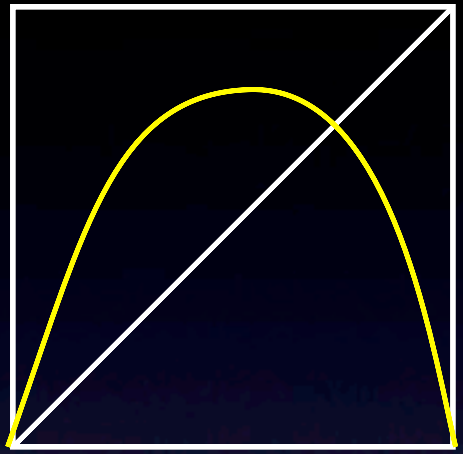
Successive construction of $\mathcal{R}f$, $\mathcal{R}^2 f$, $\ldots$, helps to understand the dynamics of f (orbits, invariant sets, rigidity, bifurcation, $\ldots$)

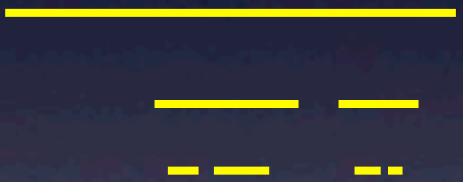


The renormalization $\mathcal{R} : f \mapsto \mathcal{R}f$ can be considered as a meta-dynamics on the space of dynamical systems of certain class.

Various Renormalizations

Feigenbaum



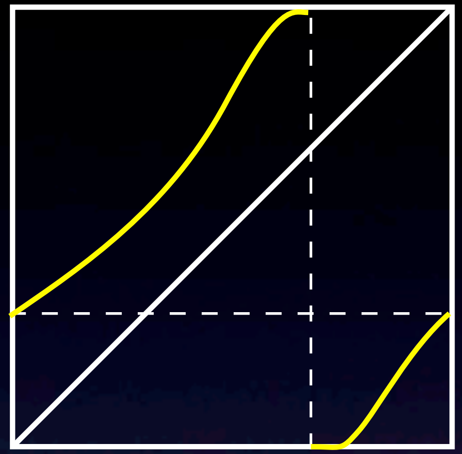


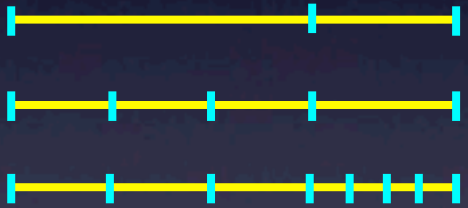
proper subintervals

-> Cantor set

Feigenbaum, Coullet-Tresser,
Lanford, H. Epstein,
Polynomial-like maps:
Douady-Hubbard, Sullivan,
McMullen, Lyubich

Circle map





partition of interval

Rand,
Katznelson-Ornstein, Khanin-Sinai,
de Melo, de Faria, Yampolsky,
A. Epstein-Yamplosky

Sector/Near-parabolic

covering by sector or croissant-like domains

gluing/identification needed to define the renormalization

Yoccoz, Perez-Marco,
Inou-S.

When a priori bounds or convergence of $\mathcal{R}^n f$ are known, it is easy to derive the properties of the original f.

Need some work to reconstruct f from $\mathcal{R}^n f$.

Local dynamics of $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots$

up to local analytic conjugacies

Trivial if $0 < |\lambda| < 1$ or $|\lambda| > 1$ (linearizable)

Assume $|\lambda| = 1$, i.e. $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R}$

Parabolic: $\alpha \in \mathbb{Q}$

Periodic or infinite dimensional moduli (Ecalle-Voronin)

Irrationally indifferent: $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

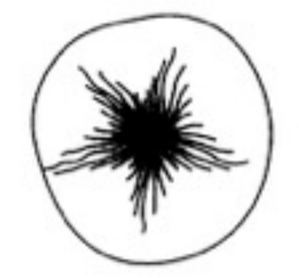
Linearizable (e.g. if α is a Brjuno number)

Siegel disk=maximal domain of linearization

Non-linearizable (e.g. if α very Liouville)

Cremer point

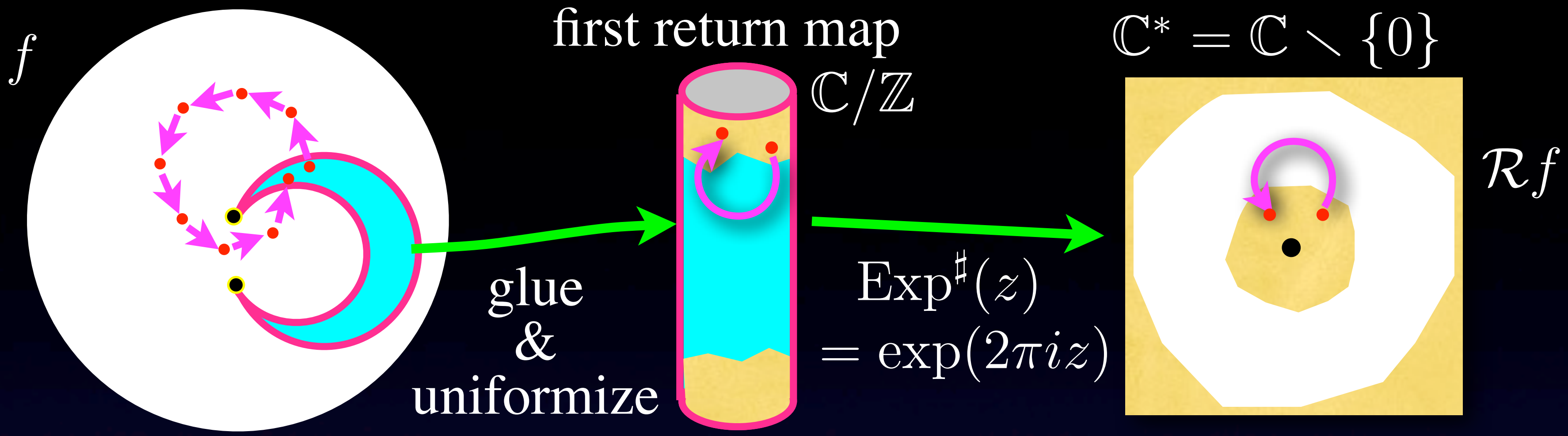
Perez Marco's *hedgehog* (local invariant set)



by Ricardo Perez-

Plan Try to describe the local dynamics near Cremer points for specific cases of quadratic polynomials with *high type* rotation numbers, via *near-parabolic renormalization* and *dynamical charts*. Construct a model for the hedgehog.

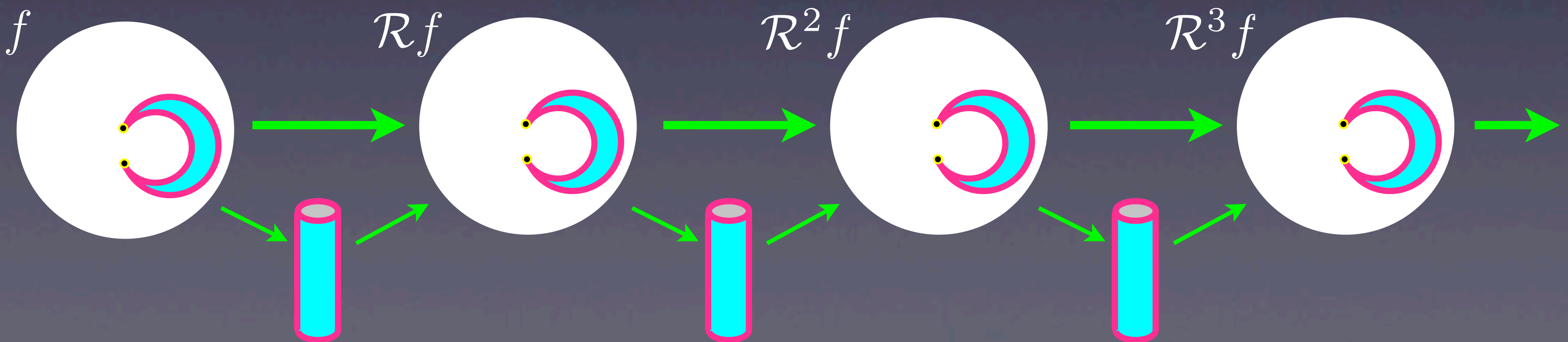
Cylinder/Near-parabolic renormalization



Hypothesis: $\exists$another fixed pt σ near 0, $\exists$croissant-shaped fundamental region, quotient is isomorphic to $\mathbb{C}/\mathbb{Z}$

$\mathcal{R}f$ can be defined when $f(z) = e^{2\pi i\alpha}z + \dots$ is a small perturbation of $z + a_2z^2 + \dots$ $(a_2 \neq 0)$ and $|\arg\alpha| < \pi/4$.

If $\mathcal{R}f, \mathcal{R}^2f, \mathcal{R}^3f, \dots$ can be defined, ...



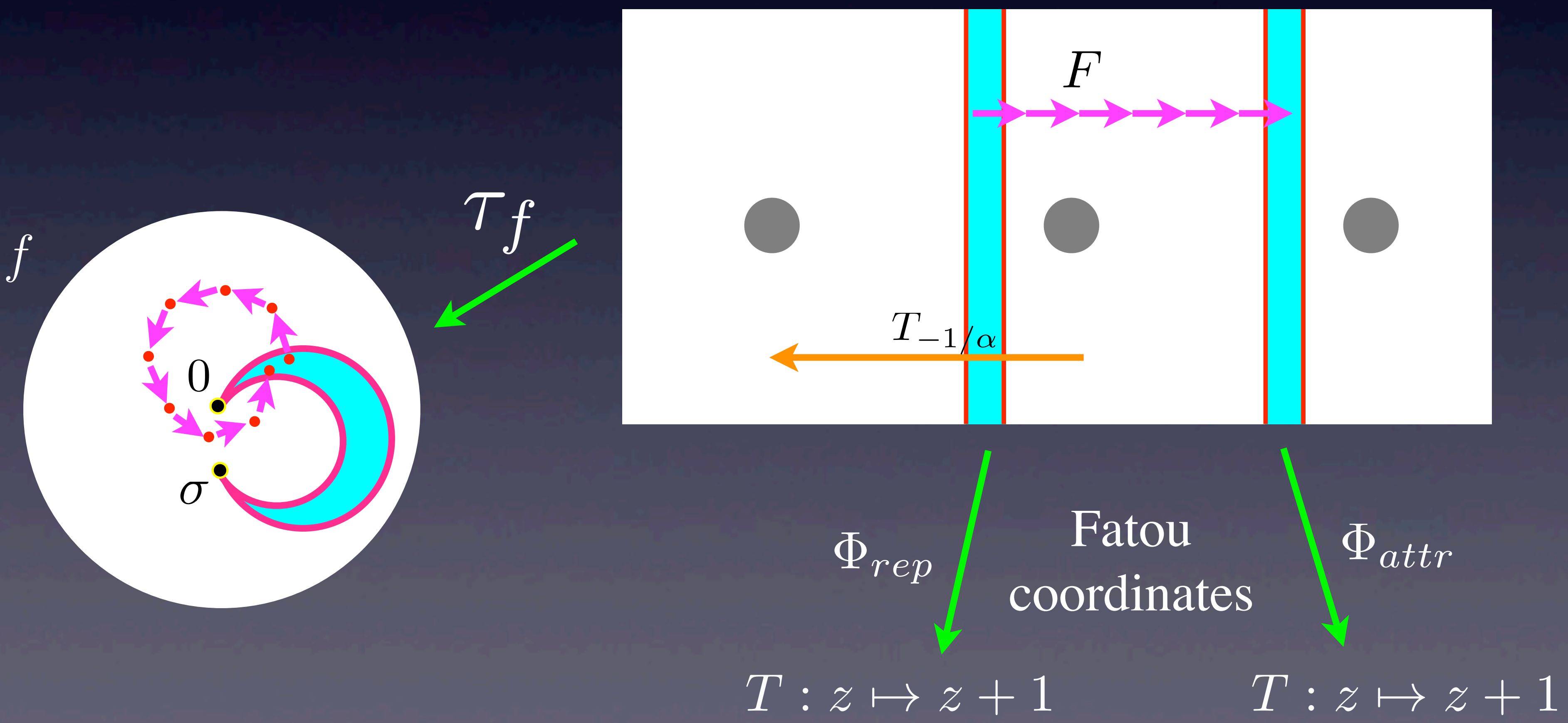
More on the construction of $\mathcal{R}f$

Suppose $f(z) = e^{2\pi i\alpha} z + \underset{\neq 0}{a_2} z^2 + O(z^2)$ has another fixed pt σ near 0.

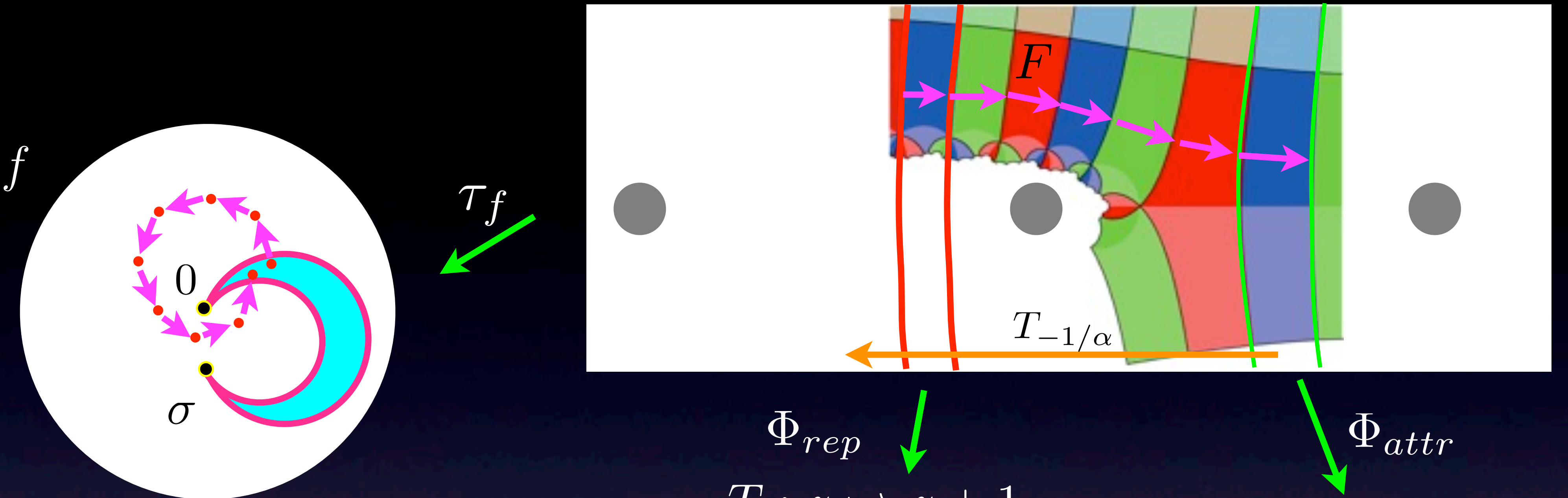
Define the pre-Fatou coordinate w by $z = \tau_f(w) = \dfrac{\sigma}{1 - e^{-2\pi i\alpha w}}$,

which is the universal covering of $\widehat{\mathbb{C}} \setminus \{0, \sigma\}$ with the deck transf. $T_{-1/\alpha} : z \mapsto z - \frac{1}{\alpha}$.

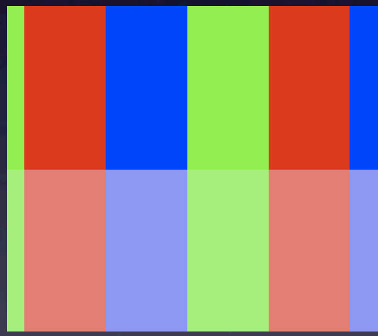
The dynamics lifts to $F(w)$ which is close to $w + 1$ outside $\cup_{n\in\mathbb{Z}} B_R$.



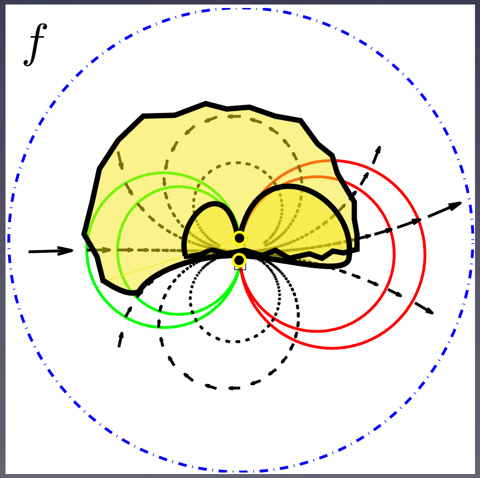
If f is from a certain class of maps containing $e^{2\pi i\alpha}z+z^2$, then ...



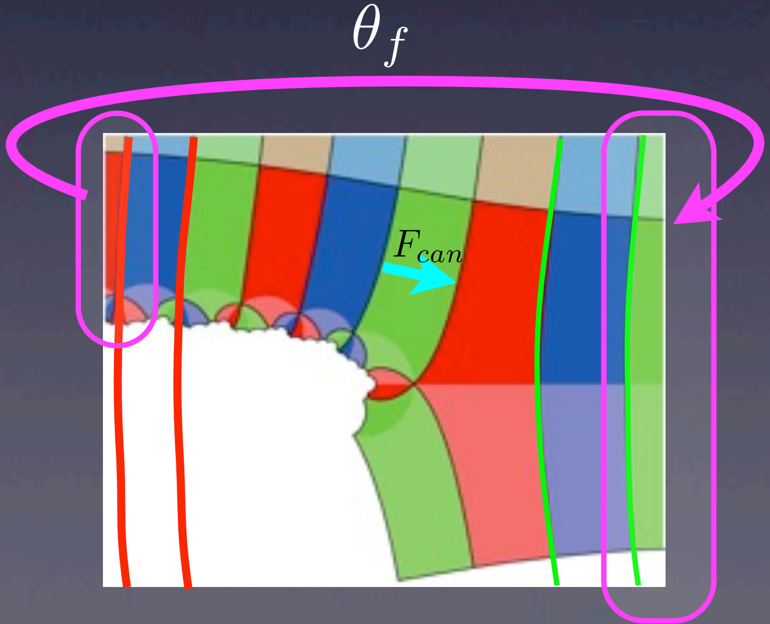
$T : z \mapsto z+1$



A Consequence from Inou-S. Theorem:
f and $\mathcal{R}^n f$ admit a "canonical representation".



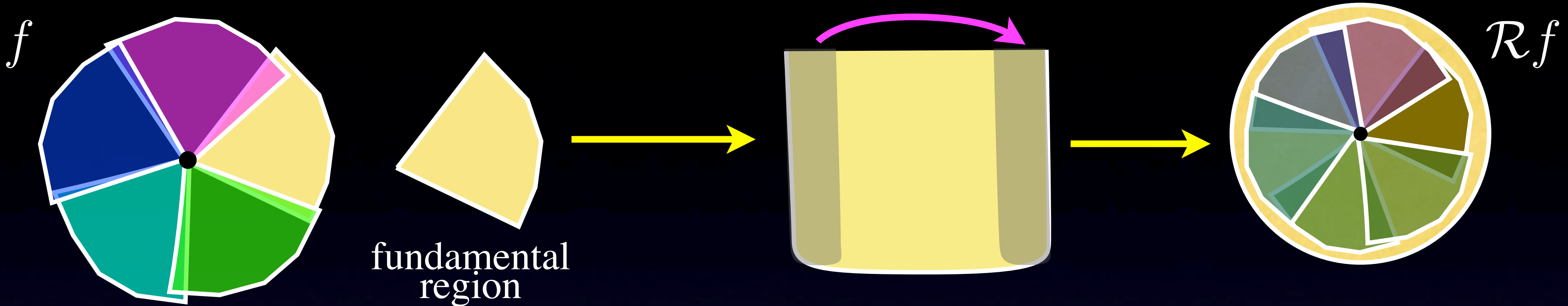
$\approx$



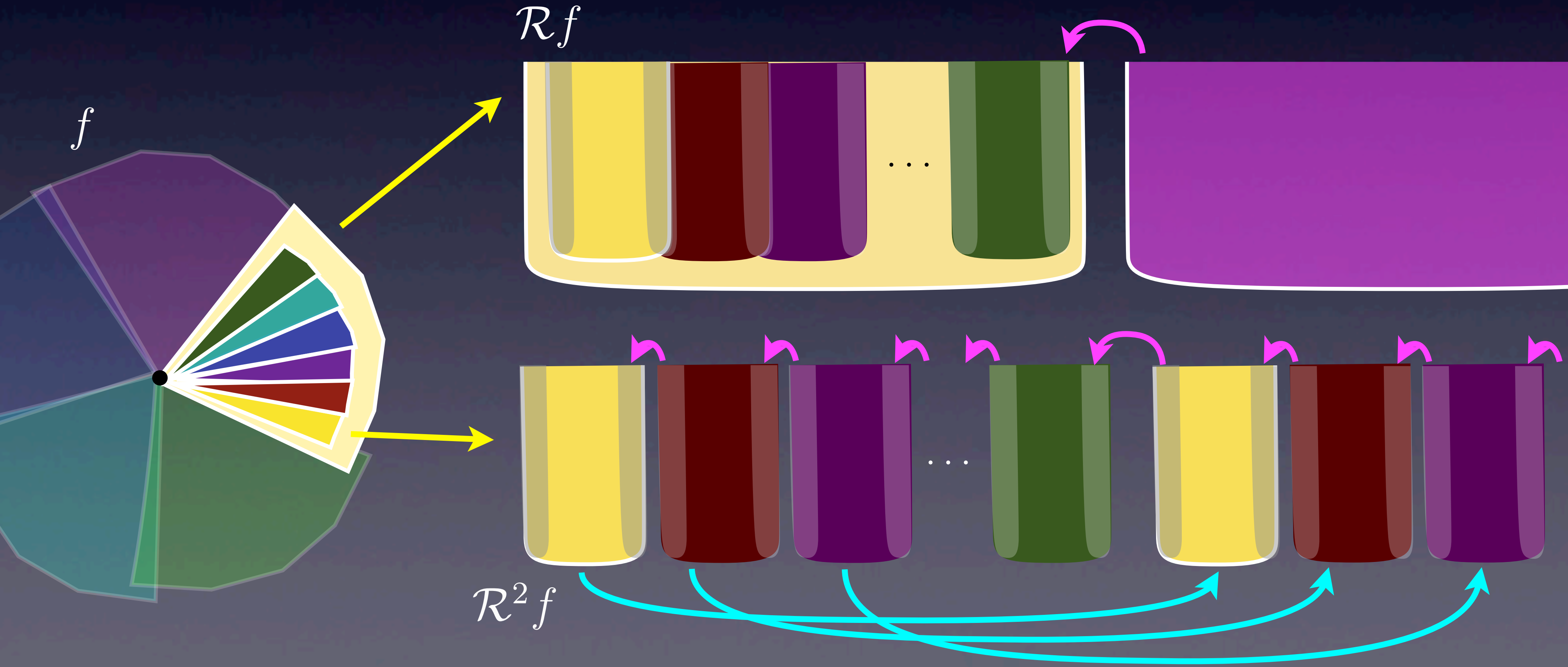
F_{can} is a *canonical* map on the *universal* pattern, NOT depending on f. Difference comes from the gluing θ_f.

Dynamical charts

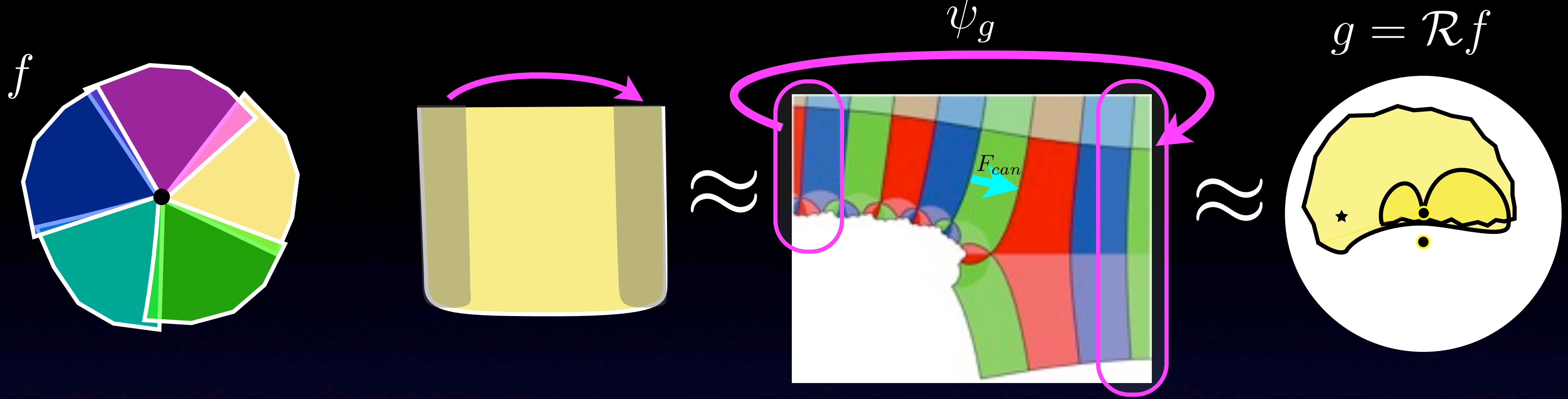
Glue model dynamics on the charts, to define/reconstruct the map

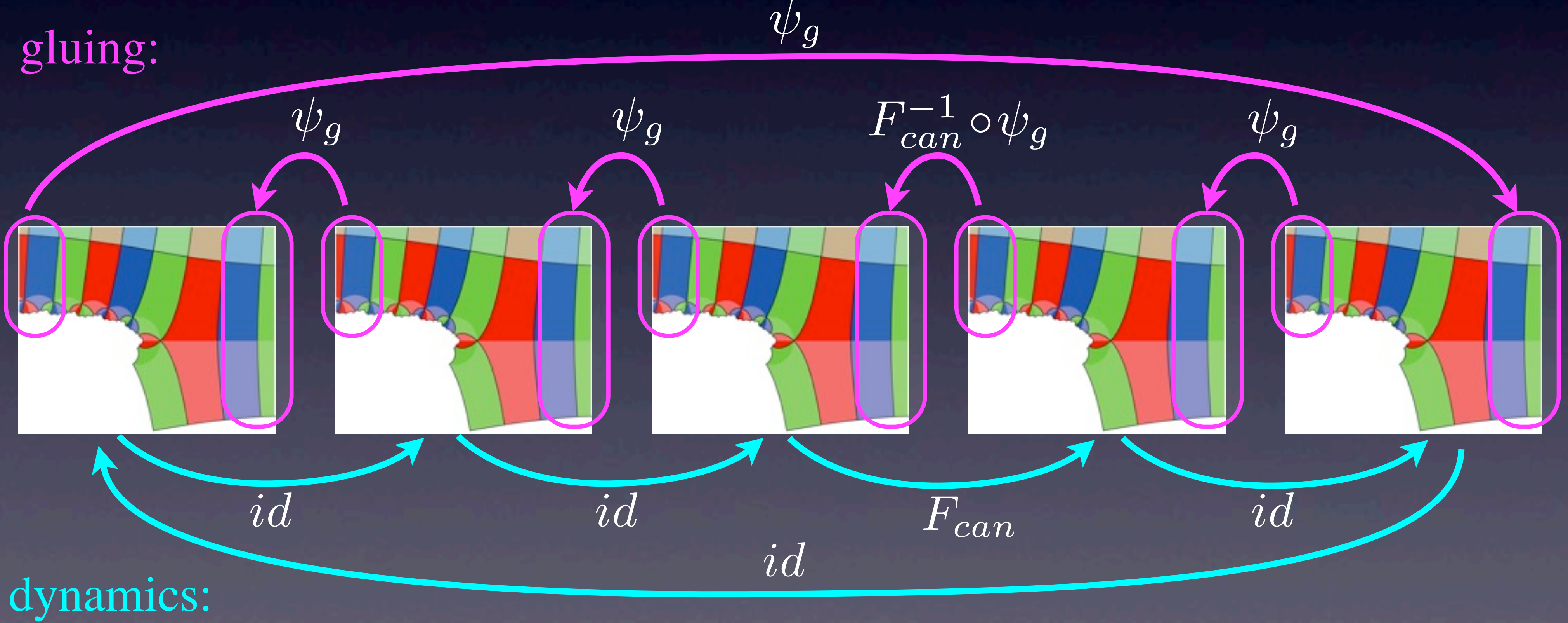


Successive renormalizations = refinements of dynamical charts



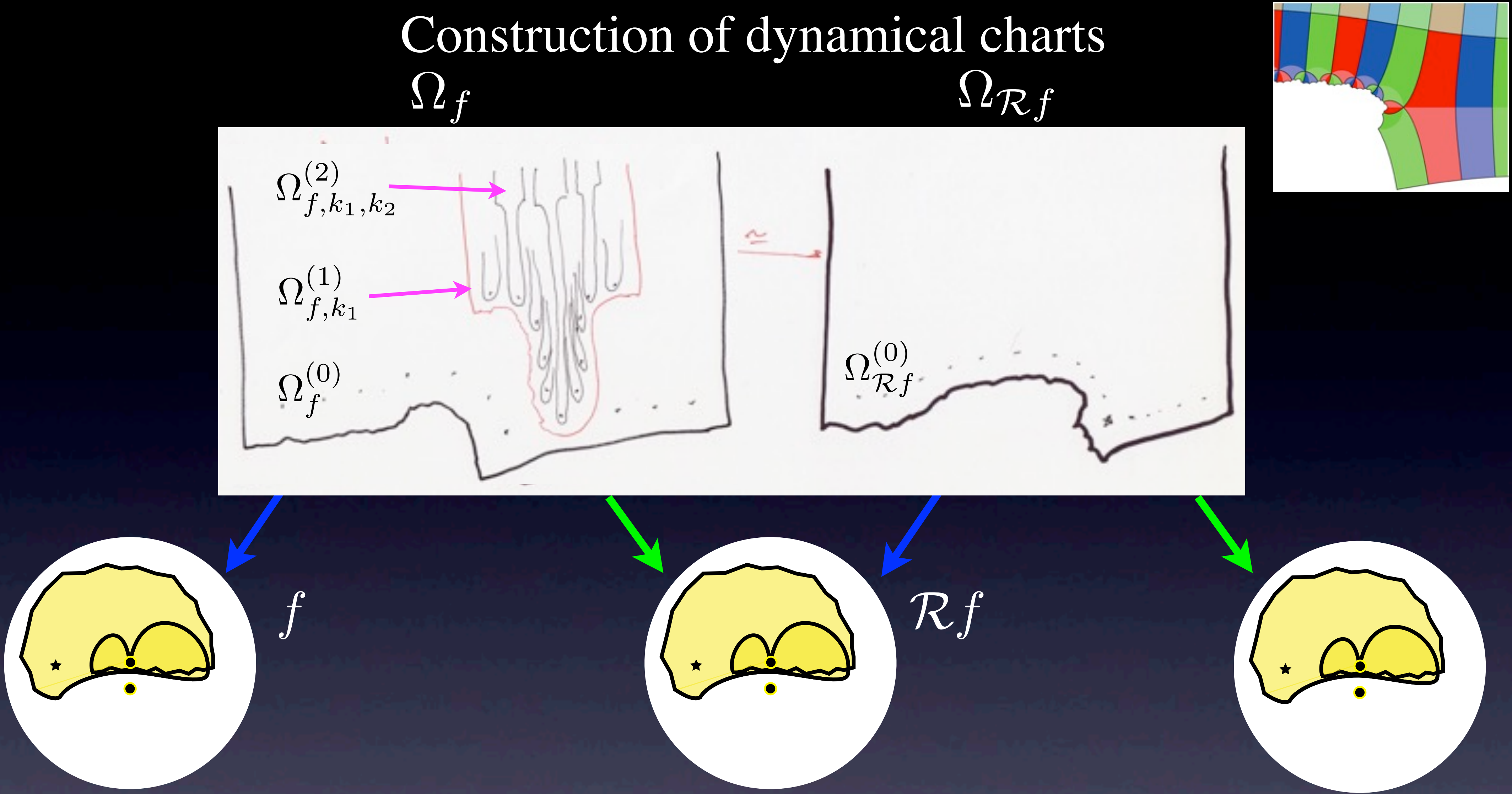
Construct 1-st dynamical chart of f from 0-th dynamical chart of $g = \mathcal{R}f$





1. well-defined after gluing 2. return map is F_{can} modulo ψ_g 3. this picture embeds into f

Construction of dynamical charts



$$\Omega^{(0)} \supset \Omega^{(1)}_{k_1} \supset \cdots \supset \Omega^{(n)}_{k_1,k_2,\ldots,k_n} \supset \Omega^{(n+1)}_{k_1,k_2,\ldots,k_n,k_{n+1}} \supset \ldots$$

each $\Omega^{(n)}_{k_1,k_2,\ldots,k_n}$ is isomorphic to truncated checkerboard pattern $\Omega_{\mathcal{R}^n f}$ they are glued via $\theta_{\mathcal{R}^n f}$

$$\Lambda_f = \bigcap_{n=0}^{\infty} \bigcup_{(k_1,\ldots,k_n)\in A_n} \Omega^{(n)}_{f,\,k_1,k_2,\ldots,k_n}$$

is an invariant set containing the critical orbit “maximal hedgehog”

Index set A_n and Denjoy odometer

$$\alpha = \pm\cfrac{1}{k_1 \pm \cfrac{1}{k_2 \pm \cfrac{1}{\ddots}}}$$

$A_n \doteqdot (\mathbb{Z}/k_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/k_n\mathbb{Z}) \subset \mathbb{Z}^n$,
where $\mathbb{Z}/k_j\mathbb{Z}$ is identified with $\left[-[\frac{k_j-1}{2}], [\frac{k_j}{2}]\right] \cap \mathbb{Z}(\subset \mathbb{Z})$

Odometer (adding machine)

$$45678 \to \underset{\text{carry}}{45679 \to 45680} \to 45681 \to \cdots \to \underset{\text{carry}}{45699 \to 45700} \to 45701 \to \ldots$$

First approximation of the combinatorics $\tilde{r}_{\alpha,n} : (\mathbb{Z}/k_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/k_n\mathbb{Z}) \circlearrowleft$

at each digit, add +1 or -1

carry to the right

carry occurs at 0 not at 9

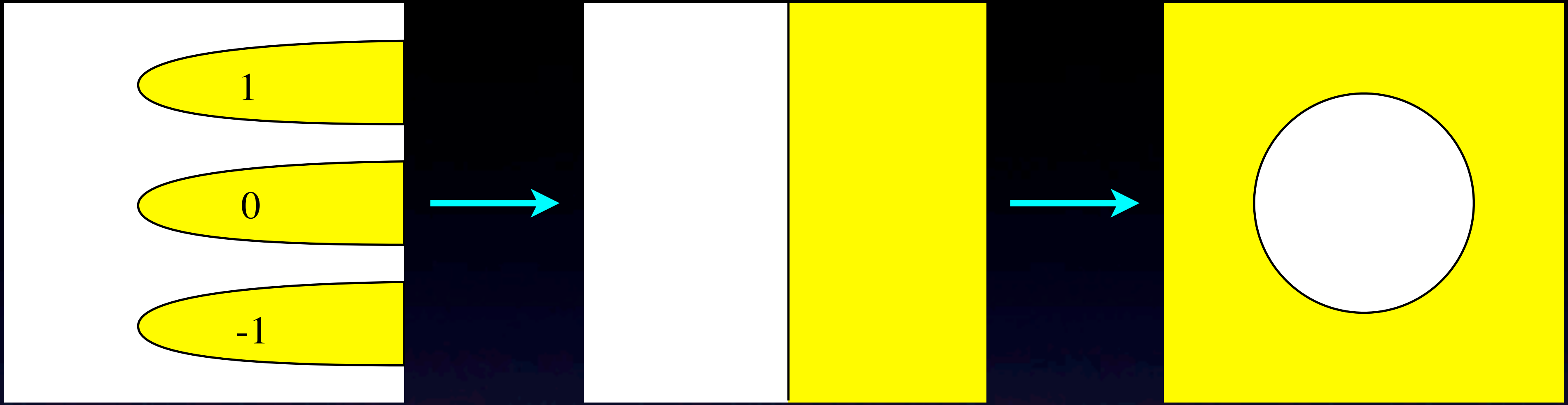
An irrational rotation is like a broken odometer.

$$45678 \to 45679 \to \underset{!}{4567A} \to 45680 \to 45681 \to \cdots \to 45689 \underset{!}{\to} 45700 \to \ldots$$

Actual combinatorics $r_{\alpha,n} : A_n \to A_n$ is modelled after the action of R_α on a partition of S^1.

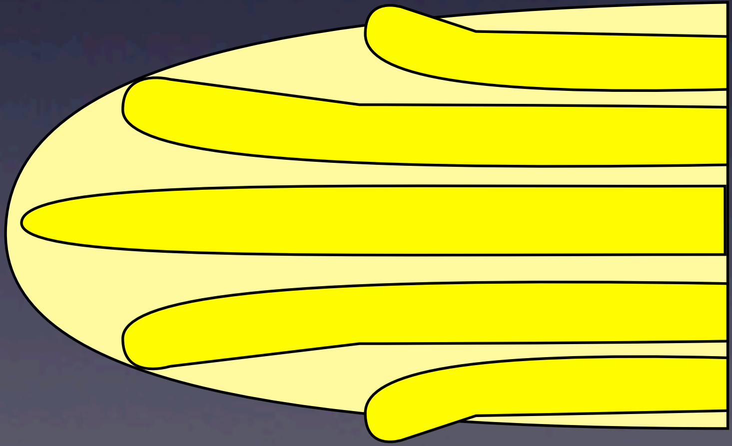
$A_\infty = \varprojlim A_n$ and $r_{\alpha,\infty} : A_\infty \to A_\infty$ is conjugate ro R_α on S^1.

Case of exponential maps $E_\lambda : z \mapsto \lambda e^z$

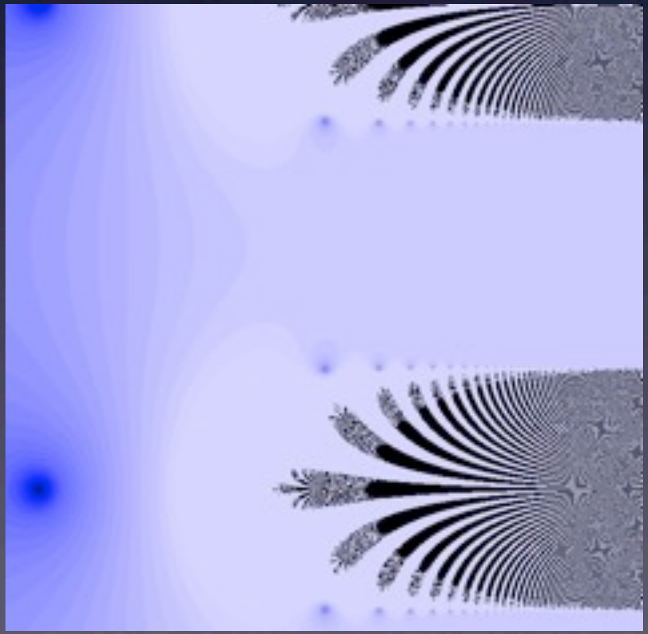


Sub-tracts
indexed by $\mathbb{Z}$

Tract



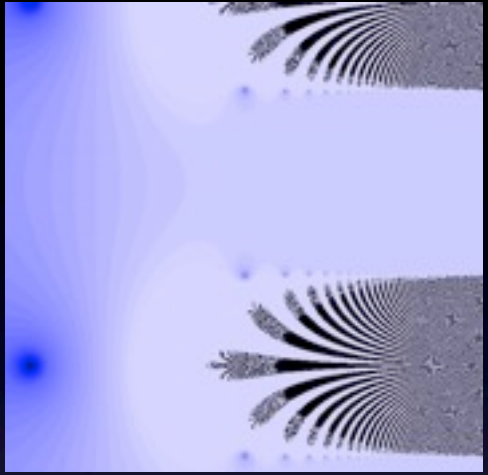
indexed by $\mathbb{Z} \times \mathbb{Z}$



Cantor bouquet: hairs indexed by $\mathbb{Z}^{\mathbb{N}}$

Rempe's straight brush model

Let $g(z) = \lambda e^z$ with $\lambda \in \mathbb{R}$ and $0 < \lambda < 1/e$. Then g has an attracting fixed pt.



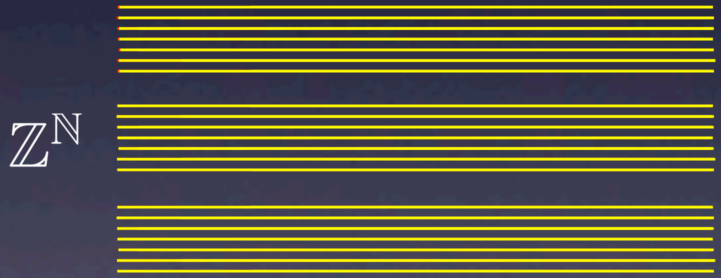
by Lasse Rempe

Itineraries $\underline{s} = (s_n)_{n\in\mathbb{N}} \in \mathbb{Z}^\mathbb{N}$ with shift σ

Model map $G : \mathbb{Z}^\mathbb{N} \times [0,\infty) \to \mathbb{Z}^\mathbb{N} \times \mathbb{R}$

$$G(\underline{s}, t) = (\sigma(\underline{s}), e^z - 1 - 2\pi|s_2|)$$

Straight brush model $\overline{X} = \{(\underline{s}, t) : \forall n \geq 0,\ proj_2(G^n(\underline{s}, t)) \geq 0\}$



Theorem (Rempe) $J(g)$ is homeomorphic to $\overline{X}$ and the dynamics g and G are conjugate.

We propose a similar model for the maximal hedgehog of irrat. indiff. fixed pts within our class.

Straight brush (Rempe) model for the hedgehogs

Denjoy odometer $A_\infty(\alpha) = \lim\limits_{\leftarrow} A_n(\alpha)$ $(\subset \mathbb{Z}^{\mathbb{N}})$,

shift $\sigma : A_\infty(\alpha_n) \to A_\infty(\alpha_{n+1})$, where $\alpha_0 = \alpha$, $\alpha_{n+1} = dist(\frac{1}{\alpha_n}, \mathbb{Z})$.

Model map $G_n : A_\infty(\alpha_n) \times [0, \infty) \to A_\infty(\alpha_{n+1}) \times \mathbb{R}$

$$G_n(\underline{s}, t) = \left(\sigma(\underline{s}), \frac{1}{\alpha_n}\log(1 + 2\pi\alpha_n e^t) - |s_2|\right)$$

modeled after
pre-Fatou coordinate

Model space

$$\overline{X} = \{(\underline{s}, t) \in A_\infty(\alpha_0) \times [0, \infty) : \forall n \geq 0,\ proj_2(G_n \circ \cdot \circ G_0(\underline{s}, t)) \geq 0\}$$

Theorem. For a map f to which Inou-S. Theorem applies and irrational number α of high type, the maximal hedgehog is homeomorphic to $\overline{X}$.

Note that $\frac{1}{\alpha}\log(1+2\pi\alpha e^t) \sim \begin{cases} 2\pi e^t & (t << \log\alpha^{-1}) \\ \alpha^{-1}(t - \log\alpha^{-1} + const) & (t >> \log\alpha^{-1}) \end{cases}$
which is related to Yoccoz' condition $\mathcal{H}$ (for global analytic conjugacy on S^1)