Singularities accumulated by regular orbits.

Adriana da Luz
(j.w. with Christian Bonatti)

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Lorenz 1963: Example of a persistent singular attractor (with periodic orbits accumulating on the singularity).

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- A Star diffeomorfism is hyperbolic and viceversa
- Not true for Flows: Splitting of singularities and periodic orbits are not compatible.

Questions and Answers

Blackboard:

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Definition

We say that a compact invariants set Λ for a vector field X is Singular hyperbolic, if the tangent space splits invariantly and continuously in one of the following ways:

- $T_\Lambda M = E^{ss} \oplus E^{cu}$ with E^{ss} contracting and E^{cu} volume expanding,
- $T_\Lambda M = E^{cs} \oplus E^{uu}$ with E^{cs} volume contracting and E^{cu} expanding.

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- It was recently proven by Yi Shi, Shaobo Gan and Lan Wen for every chain recurrent class C of X a generic star flow X, if all singularities in C have the same index, then the chain recurrent set of X is singular hyperbolic.

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In order to understand the big picture in higher dimensions, new examples need to be generated, to illustrate the different possibilities.

Theorem (j.w C.Bonatti)

There exists a compact manifold M with $\dim(M) = 5$ such that there is a $\mathcal{C}^1$ open set $\mathcal{U}^1 \subset \mathcal{X}^1(M)$ and an open set $U \subset M$ such that for any $X \in \mathcal{U}^1$ we have the following properties:

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- *U is an isolating neighborhood, and we call Λ_X the maximal invariant set of X in U.*
- *All periodic orbits in U are hyperbolic with the same index.*
- *There are 2 hyperbolic singularities σ_1 and σ_2 in Λ_X such $ind\sigma_1 = ind\sigma_2 + 1$.*
- *Both singularities are accumulated by periodic orbits and both are in the same chain recurrent class.*

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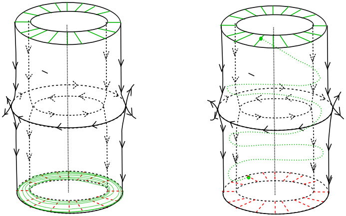

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The construction in $\mathbb{R}^3$

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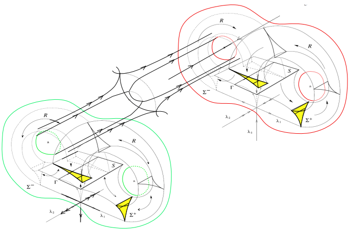

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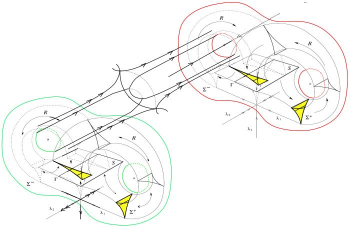

We now need to create orbits going from the lorenz atractor to the lorenz reppeler, We would not be able to do this in dimension 3.

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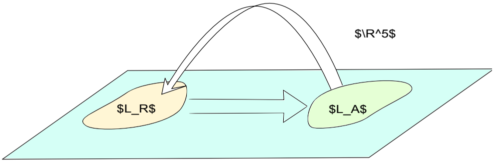

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  We call *reparametrized linear Poincaré flow* and we denote Ψ_t, the linear cocycle defined as follows: $\Psi^t(L, u) = \frac{\|D\phi^t(v)\|}{\|v\|} \cdot \psi_N^t(L, u)$ where $L \in P(M)$ is the flow direction, $u \in N$ the normal space at L and v a non vanishing vector in L.

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