Contracting Lorenz Attractor: statistical properties

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Interest

Understand the behavior of : attractors of 3-flows presenting equilibria accumulated by regular trajectories.

Main example Geometric Lorenz attractor.

First Case: Expanding

Second Case: Contracting

(1) from the topological point of view.

(2) from the statistical point of view.

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We start recalling these attractors.

Lorenz attractor

Lorenz (1963) exhibited a 3-dimensional o.d.e. whose solutions seemed to depend sensitively on the initial point.

Lorenz equations:

$$X(x,y,z) = \begin{cases} \dot{x} = 10\,(-x+y) \\ \dot{y} = 28\,x - y - xz \\ \dot{z} = -8/3\,z + xy\,, \end{cases}$$

The classical parameters are : $\alpha = 10,\ \beta = 28,\ \gamma = 8/3$.

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The flow generated by the equations above contains a volume zero attractor Λ *that is sensitive with respect to initial data.*

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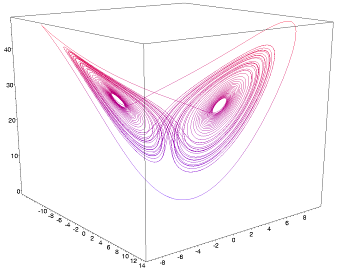
The butterfly shape of this attractor:



Solution of Lorenz Conjecture

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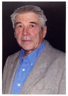
Meanwhile, it was introduced a **geometric model** for this attractor, that satisfies all all the predictions by Lorenz, that we present very briefly in the sequel.

Geometric Lorenz attractor

Introduced by GW and ABS in the seventies to model the flow generated by the Lorenz' equations.

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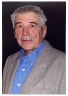
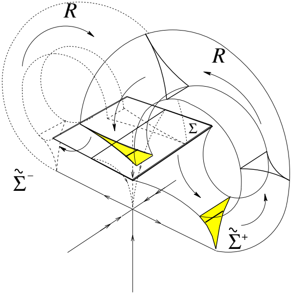
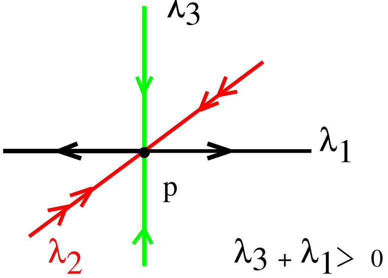




Expanding Geometric Lorenz attractor

The eigenvalues satisfy the **condition**

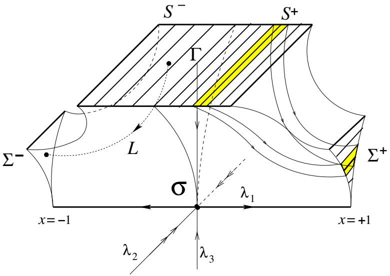
$$0 < -\lambda_3 < \lambda_1 < -\lambda_2, \quad \lambda_3 + \lambda_1 > 0.$$

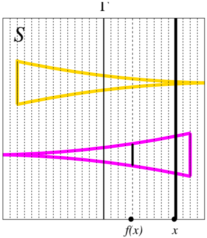


Main hypothesis

Existence of a contracting foliation $\mathcal{F}^s$ for the Poincaré map

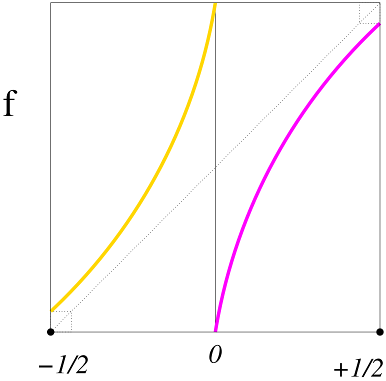
$$P : \Sigma \to \Sigma; \quad \Sigma = S^- \cup \Gamma \cup S^+$$





The quotient map

The quotient map $f : I \to I$ associated to $\mathcal{F}^s$:



Properties of $f \implies$ Lorenz conjecture

$\boxed{f \text{ is increasing, } |f'| > \sqrt{2} \text{ where it is defined and } |f'(0_\pm| = \infty.}$

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Conjecture by Lorenz follows for geometric models.

Lorenz-like attractors

  

Morales, Pacifico, Pujals

Theorem A Robust transitive sets for 3-flows are either hyperbolic or singular-hyperbolic.

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Theorem A Robust transitive sets for 3-flows are either hyperbolic or singular-hyperbolic.

Theorem B Robust transitive sets for 3-flows with equilibria are partially hyperbolic attractors or repellers.

singular-hyperbolic : partial hyperbolic, central bundle expanding area.

singular-hyperbolic attractor $\dot{\cong}$ Lorenz-like attractor

From statistical point of view

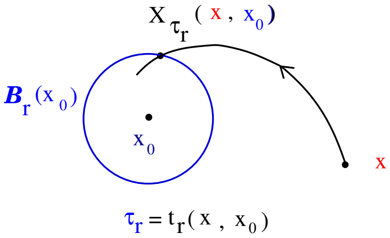
Galatolo, Pacifico

Theorem. Let F be the Poincaré map and μ the SBR measure for a Geom. Lorenz attractor. Then F has exponential decay of correlations.

Theorem. Let X^t a geometric Lorenz flow. Then for μ-almost all x, $\lim_{\mathbf{r}\to\mathbf{0}} \frac{\log \tau_{\mathbf{r}}(\mathbf{x},\mathbf{x_0})}{-\log \mathbf{r}} = \mathbf{d}_{\mu}(\mathbf{x_0}) - \mathbf{1}$.

Hitting time

Fix $x_0 \in M$ and let $B_r(x_0)$: ball with radius r at $x_0 \in \Lambda$.



Hitting time $=$ $\tau_r(x, x_0)$ is the time needed to $\mathcal{O}(x)$ enter for the first time in $B_r(x_0)$. d_μ = local dimension of μ at x_0.
When $x = x_0$, $\tau_r(x_0, x_0)$ = recurrence time.

Steps to prove Log-law for Geom. Lorenz

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3. Let $x_0 \in \Sigma$ and $\tau_{r,\Sigma}(x, x_0)$ be the time needed to $\mathcal{O}_x$ enter for the first time in $B_r(x_0) \cap \Sigma = B_{r,\Sigma}$.

   Theorem
   $\lim_{r\to 0} \frac{\log \tau_r(x,x_0)}{-\log r} = \lim_{r\to 0} \frac{\log \tau_{r,\Sigma}(x,x_0)}{-\log r} = d_{\mu_F}(x_0).$

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4. Theorem $d_\mu(x) = d_{\mu_F}(x) + 1$.

Recall: decay of correlation

$F : \Sigma \to \Sigma$: a first return map to Σ, a cross section to X^t.

The map F has exponential decay of correlations:

$\exists\ C > 0,\ \lambda < 1$ such that $\quad \forall\ n \geq 1 \quad$ it holds

$$\boxed{\left| \int g(F^n(x)) \cdot f(x) d\mu - \int f(x) d\mu \cdot \int g(x) d\mu \right| \leq C \lambda^n.}$$

Main difficult: F has exponential decay

Ingredient:

The Wasserstein-Kantorovich distance defined as follows:

Given two probabilities on M, μ_1 and μ_2, the W-K-distance is

$$\boxed{W(\mu_1, \mu_2) = \sup_{g \in Lip_1(M)} \left(\left| \int_M g d\mu_1 - \int_M g d\mu_2 \right| \right)}$$

$Lip_1(M)$: the space of 1-Lipschitz maps on M.

W-K distance versus decay

Proposition 1. (decay in function of distance) Let $\mu_1 \ll \mu$ and $d\mu_1 = f(x)d\mu$. Then, for $g \in Lip_1(M)$ we have

$$\boxed{|\mathcal{C}_n(g, f)| \leq L(g) \cdot \|f\|_1 \cdot W((F^*)^n(\mu_1), \mu).}$$

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Proposition 2. (distance in function of decay) Assume that for each $f \in L^1(\mu)$ and $g \in Lip_1(M)$ it holds:

$$|\mathcal{C}_n(g,f)| \leq C \cdot \|g\|_{Lip_1(M)} \cdot \|f\|_{L^1(\mu)} \cdot \Phi(n).$$

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Then, taking $d\mu_1 = \frac{f}{\|f\|_1}$ with $\int f(x)d\mu = 1$ we get

$$\boxed{W((F^*)^n(\mu_1), \mu) \leq 2 \cdot C \cdot \Phi(n).}$$

W-K dist. versus disintegration

Proposition 3. Let μ^1 and μ^2 be invariant for (F, Σ) s.t.

- $\mu^1(A) = \int \mu^1_\gamma(A \cap \gamma) d\mu^1_x$,
- $\mu^2(A) = \int \mu^2_\gamma(A \cap \gamma) d\mu^2_x$,

with μ^i_x having bounded variation density. Also,

(1) for each $\gamma \in \mathcal{F}^s$, $W_1(\mu^1_\gamma, \mu^2_\gamma) \leq \epsilon$,

(2) $\sup_{\|h\|_\infty} \left| \int h d\mu^1_x - \int h d\mu^2_x \right| \leq \delta$.

Then

$$\boxed{W(\mu^1, \mu^2) \leq \epsilon + \delta.}$$

(Σ, F, μ) is fastly mixing

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Strategy:

Combine the three propositions above to deduce exponentially decay of correlations.

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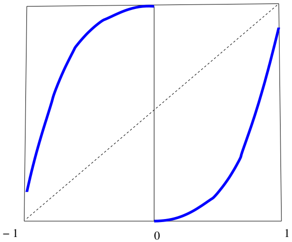
1-dimensional Rovella map

The 1-dimensional Rovella map f_0 :



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On opposition to the robustness of a Lorenz attractor, the Rovella attractor is not robust .

But, it persists in a measure theoretical sense: there exists a one-parameter family of positive Lebesgue measure of C^3 close vector fields to X_0, the starting vector field, which have a transitive non-hyperbolic attractor.

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Strategy in this case

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This time $F : \Sigma \to \Sigma$ is a skew-product with non-expanding base map $f : I \to I$.

Thus, we have to improve the arguments. We follow a general principle: the statistical properties of a skew product are obtained from the statistical properties of the base map. To do so we proceed as follows.

Convergence to equilibrium

Definition We say that (f, μ) has exponential convergence to equilibrium with respect to a reference measure ν and norms $\|.\|_a$ and $\|.\|_b$, if there are $C, \Lambda \in \mathbb{R}^+,\ \Lambda < 1$ such that for $f \in L^1(\nu), g \in L^1(\mu)$ and all $n \geq 1$ it holds

$$\boxed{\left|\int f \cdot (g \circ T^n)\, d\nu - \int g\, d\mu \int f\, d\nu\right| \leq C\Lambda^n \cdot \|g\|_a \cdot \|f\|_b.}$$

Definition Let $Q = I \times I$ and $f : Q \to \mathbb{R}$ is integrable, we denote by $\pi(f) : I \to \mathbb{R}$ the function

$$\pi(f) : x \mapsto \int_I f(x,t)\ dt.$$

Skew product, base having exp-conv to equilibrium

Theorem Let $F : Q \circlearrowleft$, $F(x,y) = (T(x), G(x,y))$. Let μ be F-invariant measure with bsolutely continuous marginal μ_T on the x-axis which, moreover, is T-invariant. Suppose that

1. (T, μ_T) has exponential decay of correlations with respect to the norm $\|\cdot\|_\infty$ and to a norm denoted by $\|\cdot\|_{\_}$.
2. T: nonsingular resp to Lesbegue, piecewise continuous, monotonic: $\cup I_i = I$, T: homeo onto its image.
3. G is λ-Lipschitz in y with $\lambda < 1$.

Then (F, μ) has exponential convergence to equilibrium with respect to $\nu = \mu_T \times m$ (the product of the a.c.i.m of T and the Lebesgue measure).

Exp-conv to equilibrium

In the following sense:

There are $C, \Lambda \in \mathbb{R}^+,\ \Lambda < 1$ such that

$$\left| \int f \cdot (g \circ F^n)\, d\nu - \int g\, d\mu \int f\, d\nu \right| \leq$$

$$C\Lambda^n \cdot \|g\|_{\updownarrow lip} \cdot (||\pi(f)||_{\text{-}} + ||f||_1)$$

for each $f \geq 0$.

Conv equilibrium versus decay correlation

Let $F : Q \circlearrowleft,\ F(x,y) = (T(x), G(x,y))$, μ a F-invariant probability measure with absolutely continuous T-invariant marginal μ_T on the x-axis and satisfying

1. (T, μ_T) has exponential convergence to equilibrium with respect to the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\_}$;
2. T is nonsingular with respect Lesbegue measure, piecewise continuous and monotonic, $\exists\ \{I_i\}_{i=1,\dots,m}$, $\cup I_i = I$ so that I_i, T is an homeo onto its image.
3. F is a uniform contraction on each vertical leaf.

Then F has exp decay of correlations:

$$\left| \int f \cdot (g \circ F^n)\ d\mu - \int g\ d\mu \int f\ d\mu \right| \leq$$

$$C_2 \Lambda^n \|g\|_{\updownarrow lip} (\|f\|_{\updownarrow lip} + \|\pi(f)\|_{\_} + \|f\|_{\square}).$$

Main Difficulties

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The base map is not piecewise increasing anymore.

($\star$) The main technical problem is transform the information we have about the base map (exponential decay with respect to Holder and L_∞ observables) into information about decay of correlation respect to generalized bounded variation observables.

The Rovella skew product $F : Q \circlearrowleft$

(a) $F(x, y) = (T(x), G(x, y))$ (preserves vertical foliation),

(b) $\exists\ c \in I,\ k \geq 0$ s.t. if $c \notin [x_1, x_2]$ then $\forall\ y \in I : |G(x_1, y) - G(x_2, y)| \leq k \cdot |x_1 - x_2|$,

(c) $F|_\gamma$ is λ-Lipschitz, $\lambda < 1$ on each leaf γ,

(d) T is onto, $x = 0$ is a discontinuity, piecewise C^3, two branches, $\mathcal{O}(T'(0)) = s - 1 > 0$, $T'(x) > 0\ \forall\ x \neq 0$,

(e) $\max_{x>0} T'(x) = T'(1)$ and $\max_{x<0} T'(x) = T'(-1)$,

(f) 1 and -1 are pre-periodic repelling,

(g) T has negative Schwarzian derivative: $S(T) < \alpha < 0$.

Main Theorem

Theorem $F : Q \circlearrowleft$ satisfying properties (a)–(g) above. Then *the unique SBR measure* μ_F has exponential decay of correlation respect to $\mathbb{B}(1,\alpha)$ and L^∞ observables, that is, there are $C, \Lambda \in \mathbb{R}^+,\ \Lambda < 1$, such that

$$\left| \int f \cdot (g \circ F^n)\, d\mu_F - \int g\ d\mu_F \int f\ d\mu_F \right| \leq C\Lambda^n \|g\|_{L^\infty} \|f\|_{\mathbb{B}(1,\alpha)},$$

$\mathbb{B}(1,\alpha)$: Banach space of generalized bounded variation maps.

Finally

$F : Q \circlearrowleft$ has exponential decay $\Rightarrow$ log law for the hitting time.

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Many thanks

Grazie mille