“You, my forest and water! One swerves, while the other shall spout
Through your body like draught; one declares, while the first has a doubt.”
J. Brodsky

Ты, мой лес и вода, кто объедет, а кто, как сквозняк,
проникает в тебя, кто глаголет, а кто обиняк...
И. Бродский

(Cries and whispers in windtree forests)

Diffusion of wind-tree billiards and Lyapunov exponents of the Hodge bundle

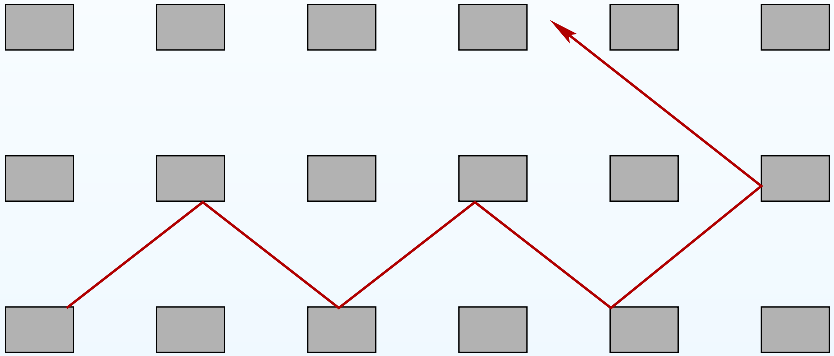
Anton Zorich (joint work with Vincent Delecroix)

School and Conference on Dynamical Systems
ICTP, Trieste, July 2015

0. Model problem: diffusion in a periodic billiard

Diffusion in a periodic billiard (Ehrenfest "Windtree model")

Consider a billiard on the plane with $\mathbb{Z}^2$-periodic rectangular obstacles.



Old Theorem (V. Delecroix, P. Hubert, S. Lelièvre, 2014). *For all parameters of the obstacle, for almost all initial directions, and for any starting point, the billiard trajectory spreads in the plane with the speed $\sim t^{2/3}$. That is,*

$$\lim_{t\to+\infty} \log \text{(diameter of trajectory of length } t)/\log t = 2/3.$$

The diffusion rate $\frac{2}{3}$ is given by the Lyapunov exponent of certain renormalizing dynamical system associated to the initial one.

Remark. Changing the height and the width of the obstacle we get quite different billiards, but this does not change the diffusion rate!

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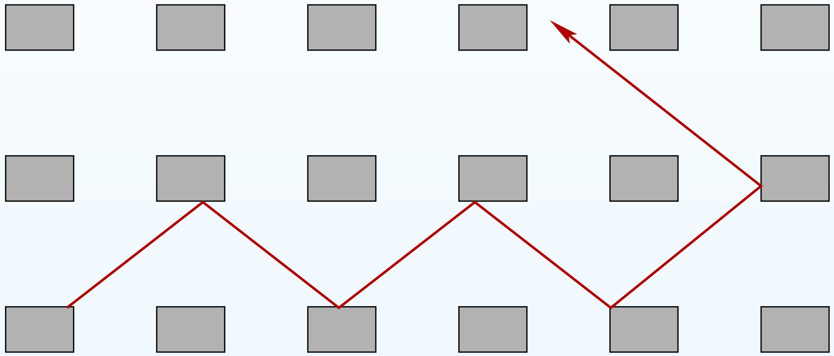

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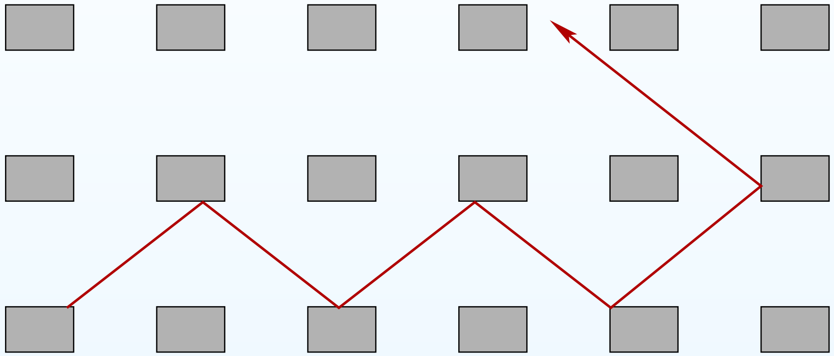

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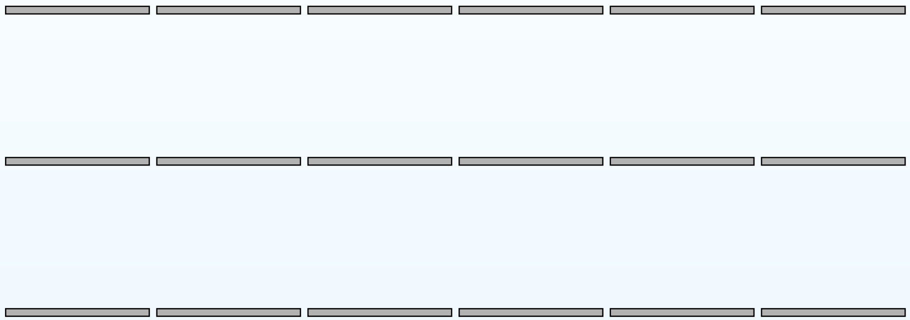

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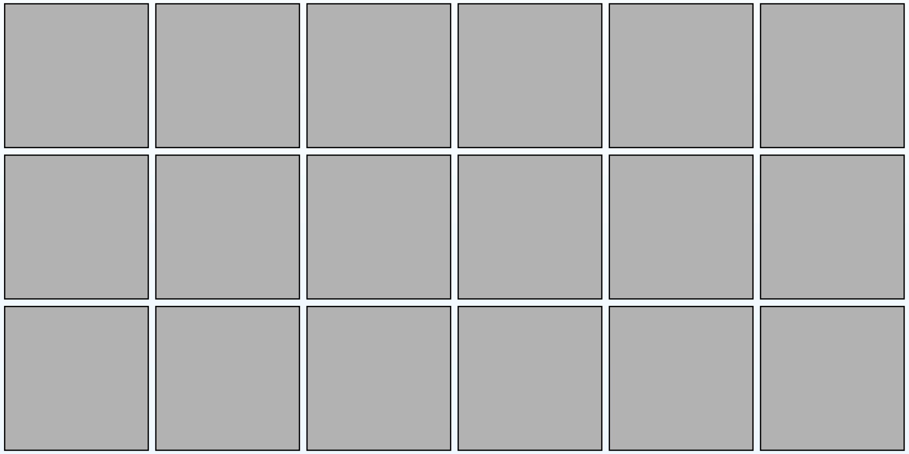

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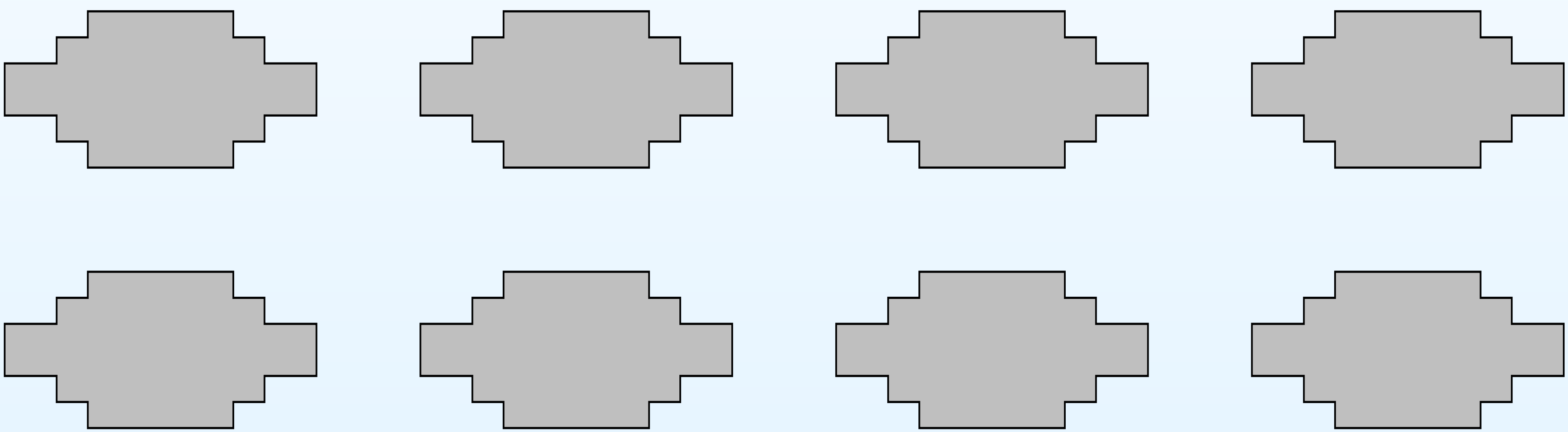

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Question of J.-C. Yoccoz

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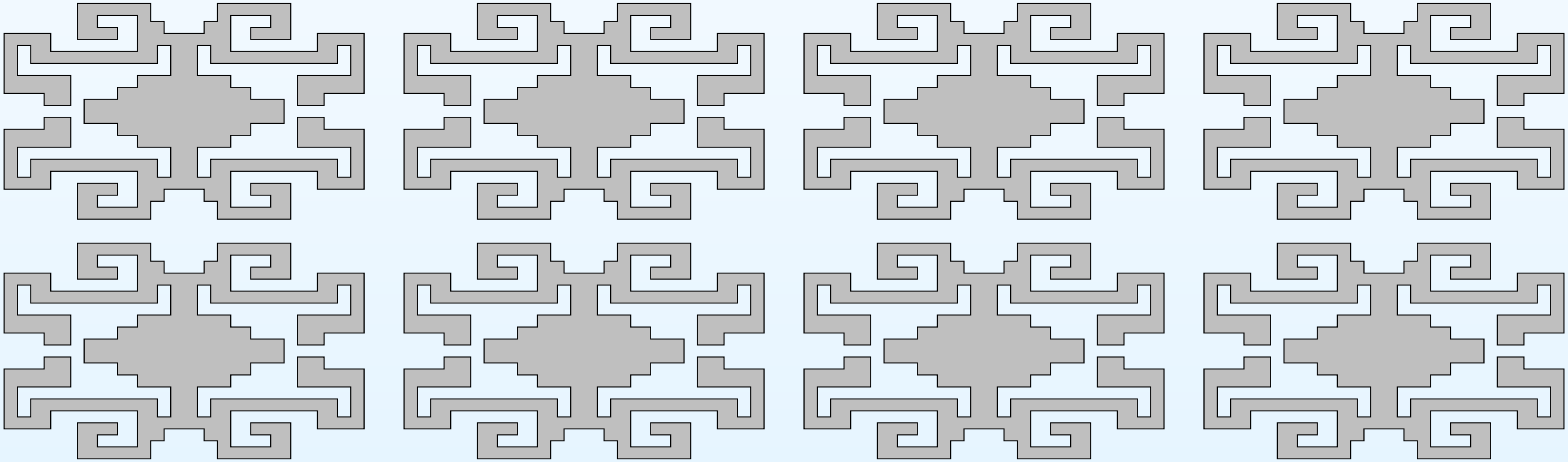

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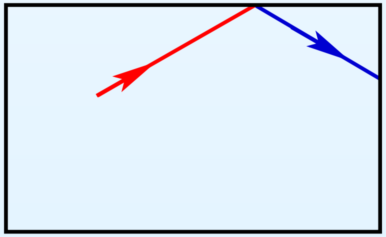
From a billiard to a surface foliation

Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of four directions defining four types of copies of the billiard table. Copies of the same type are related by a parallel translation.



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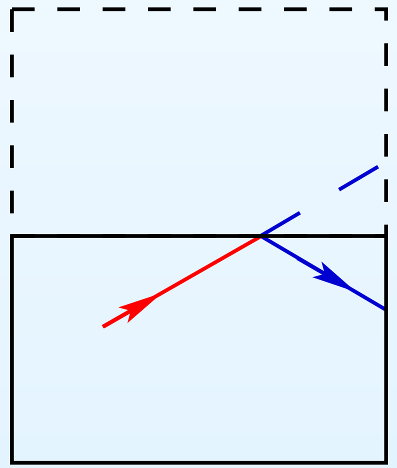

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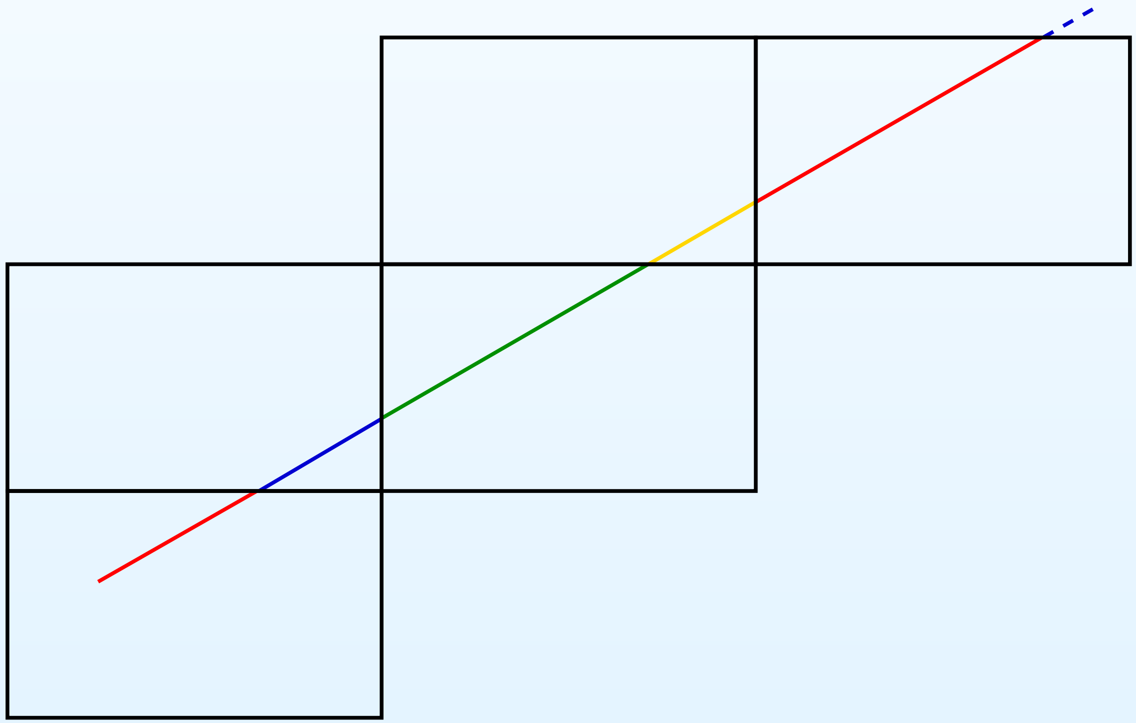

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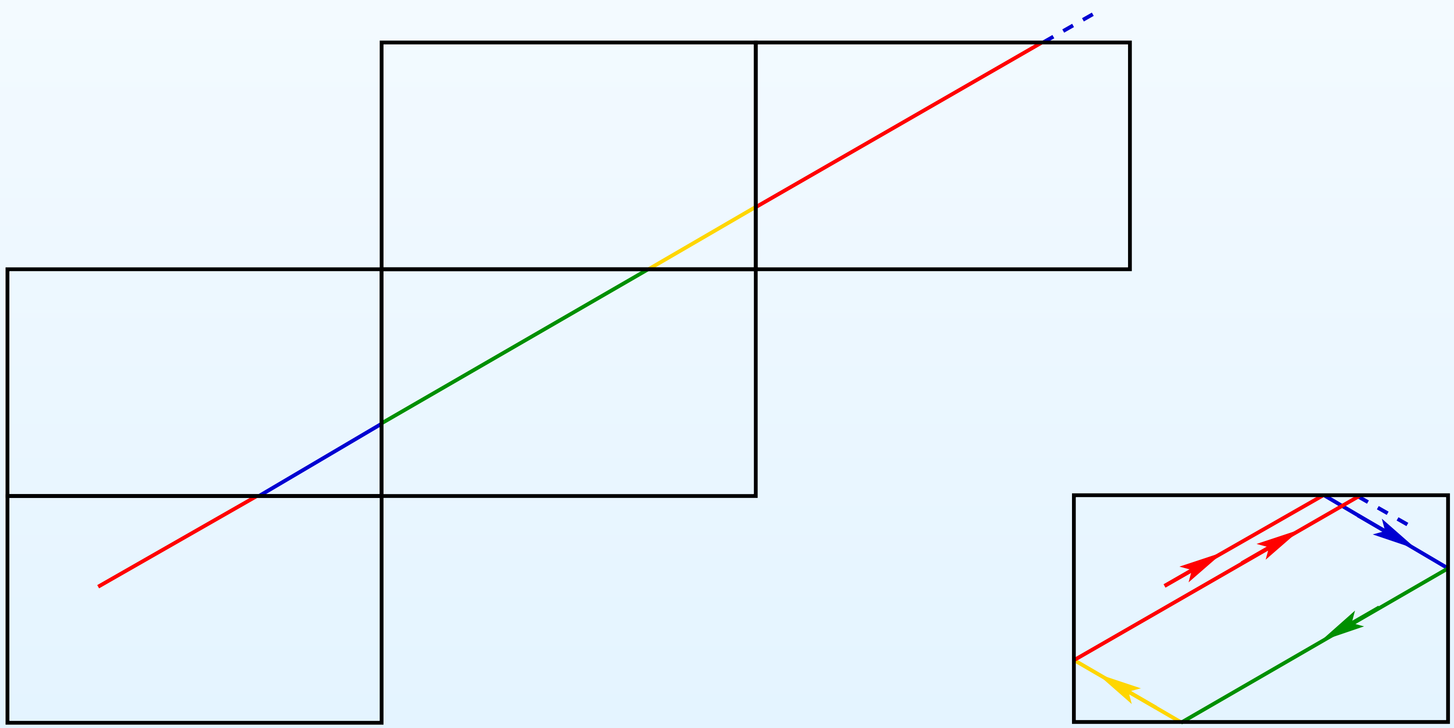

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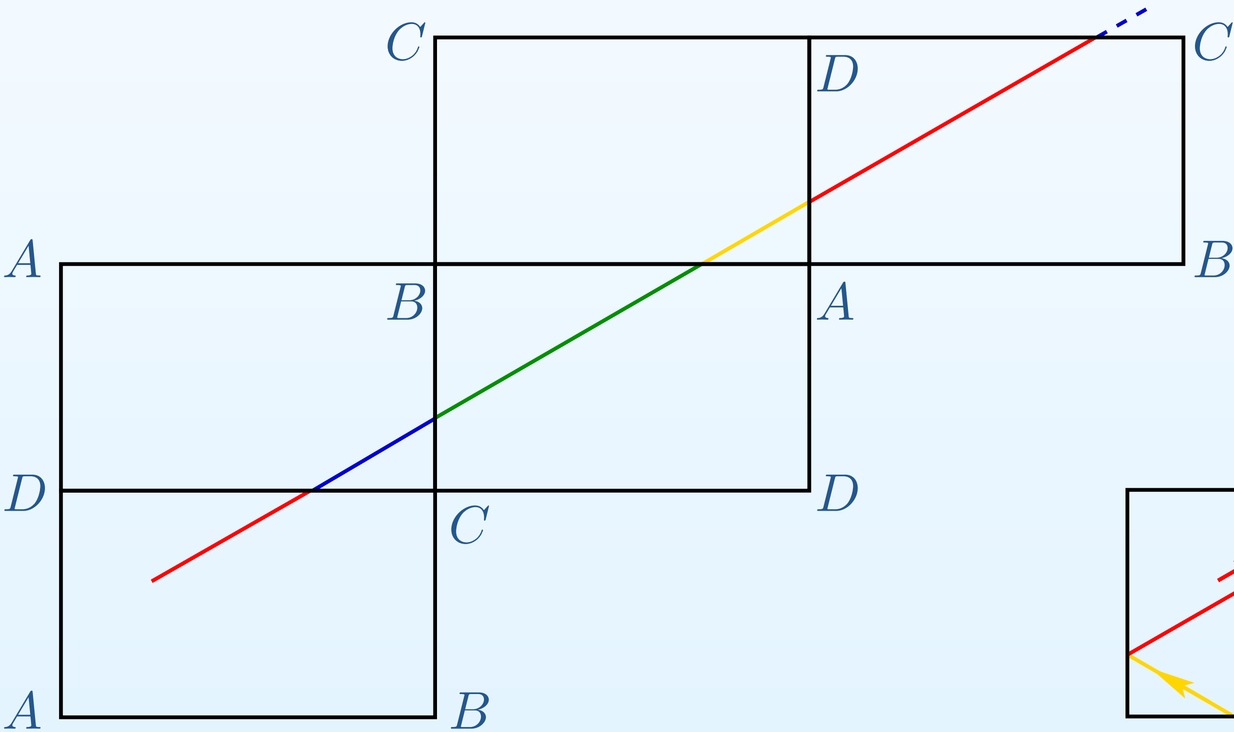

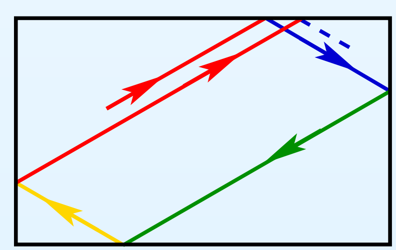

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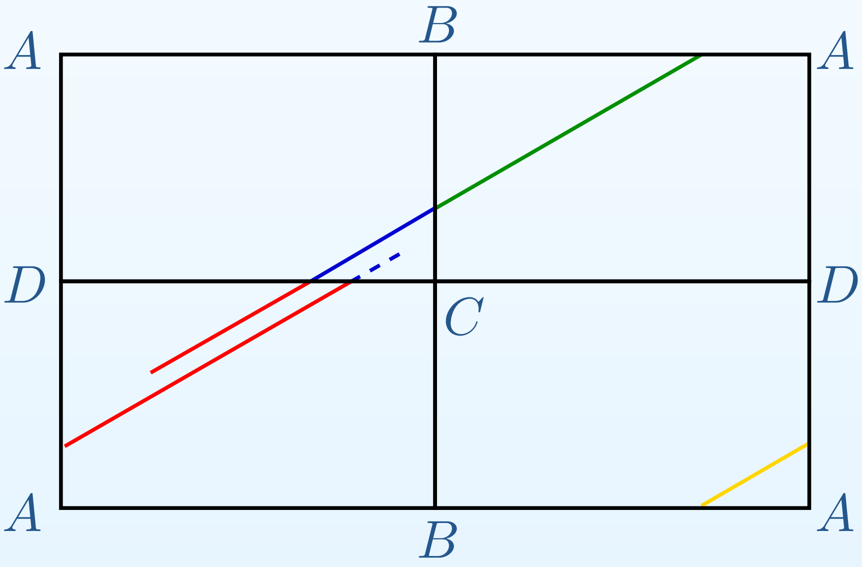
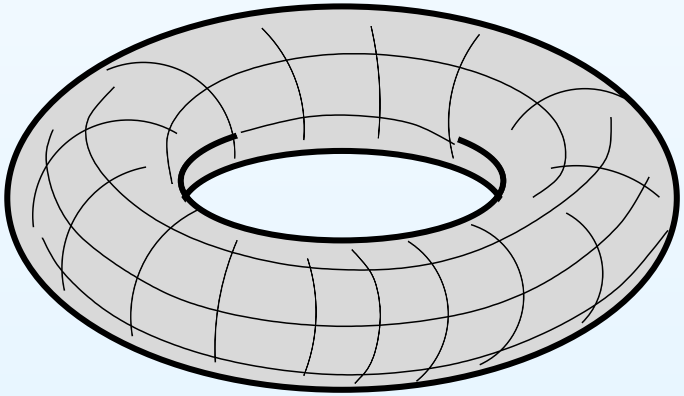
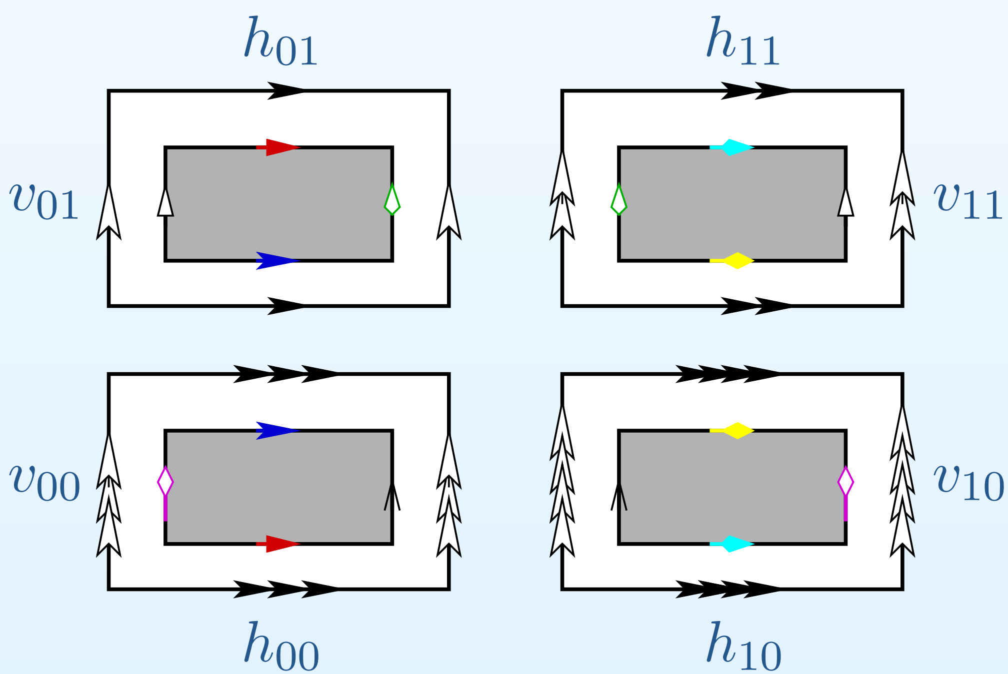




Identifying the equivalent patterns by a parallel translation we obtain a torus; the billiard trajectory unfolds to a "straight line" on the corresponding torus.

From the windtree billiard to a surface foliation

Similarly, taking four copies of our $\mathbb{Z}^2$-periodic windtree billiard we can unfold it to a foliation on a $\mathbb{Z}^2$-periodic surface. Taking a quotient over $\mathbb{Z}^2$ we get a compact surface endowed with a measured foliation. Vertical and horizontal displacement (and thus, the diffusion) of the billiard trajectories is described by the intersection numbers $c(t) \circ v$ and $c(t) \circ h$ of the cycle $c(t)$ obtained by closing up a long piece of leaf with the cycles $h = h_{00} + h_{10} - h_{01} - h_{11}$ and $v = v_{00} - v_{10} + v_{01} - v_{11}$.

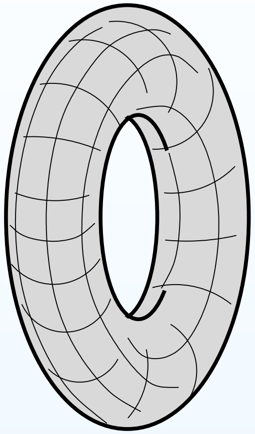


1. Teichmüller dynamics (following ideas of B. Thurston)

Diffeomorphisms of surfaces

Observation 1. *Surfaces can wrap around themselves.*

Cut a torus along a horizontal circle.



Diffeomorphisms of surfaces

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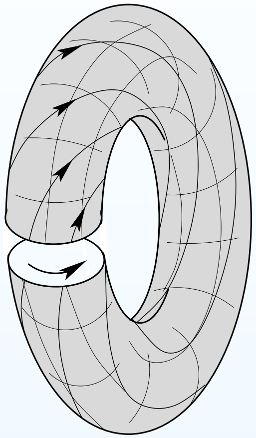
Twist de Dehn twists progressively horizontal circles up to a complete turn on the opposite boundary component of the cylinder and then identifies the components.



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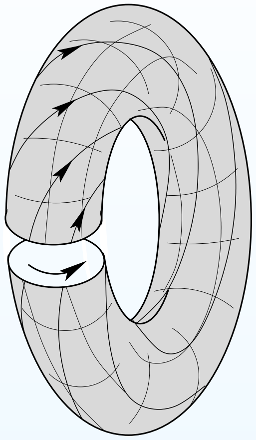

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\hat{f}_h} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2 & \xrightarrow[f_h]{} & \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \end{array}$$

Dehn twist corresponds to the linear map $\hat{f}_h : \mathbb{R}^2 \to \mathbb{R}^2$ with the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

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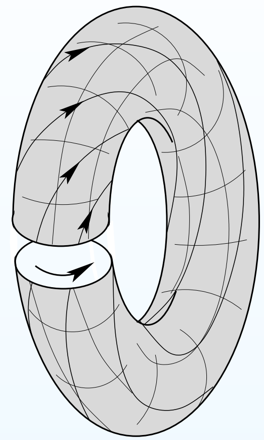

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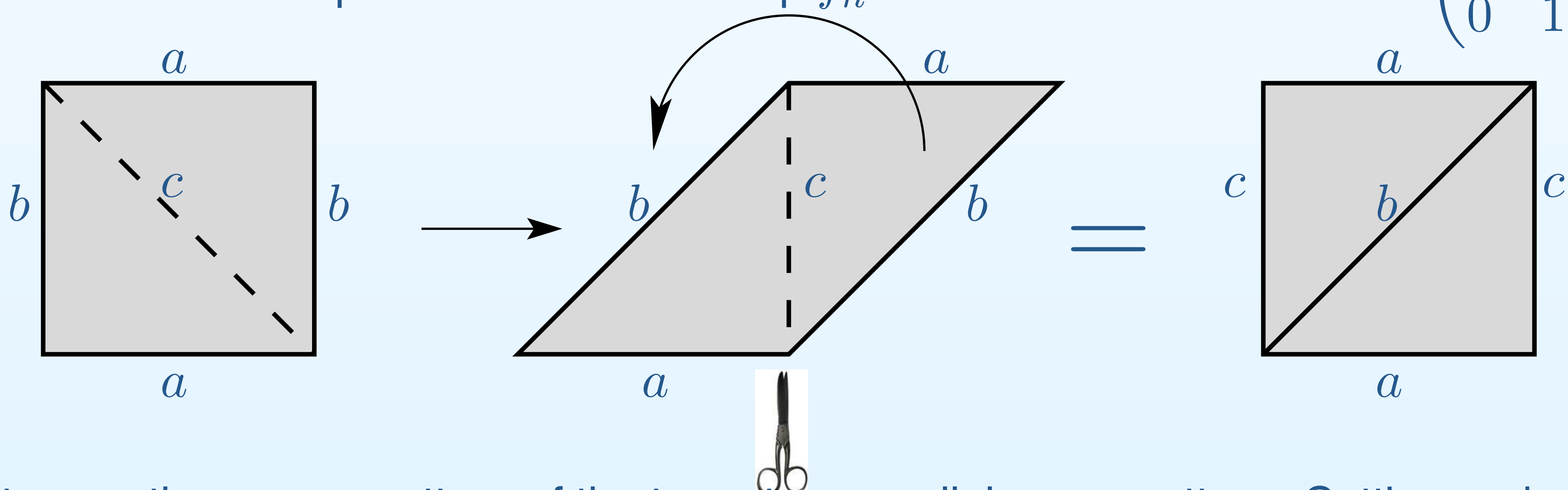
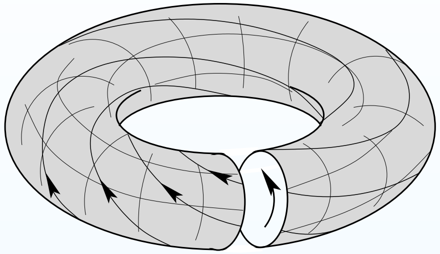
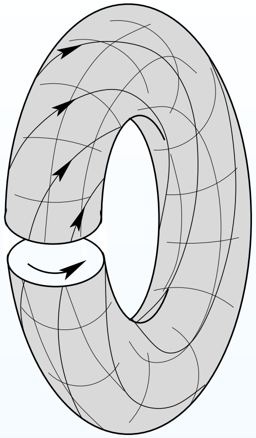


It maps the square pattern of the torus to a parallelogram pattern. Cutting and pasting appropriately we can transform the new pattern to the initial square one.

Pseudo-Anosov diffeomorphisms

Consider a composition of two Dehn twists

$$g = f_v \circ f_h =$$  $\circ$ 

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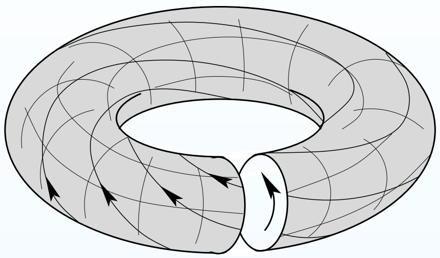
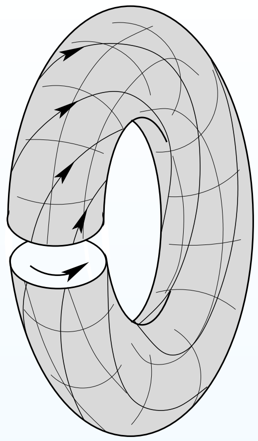
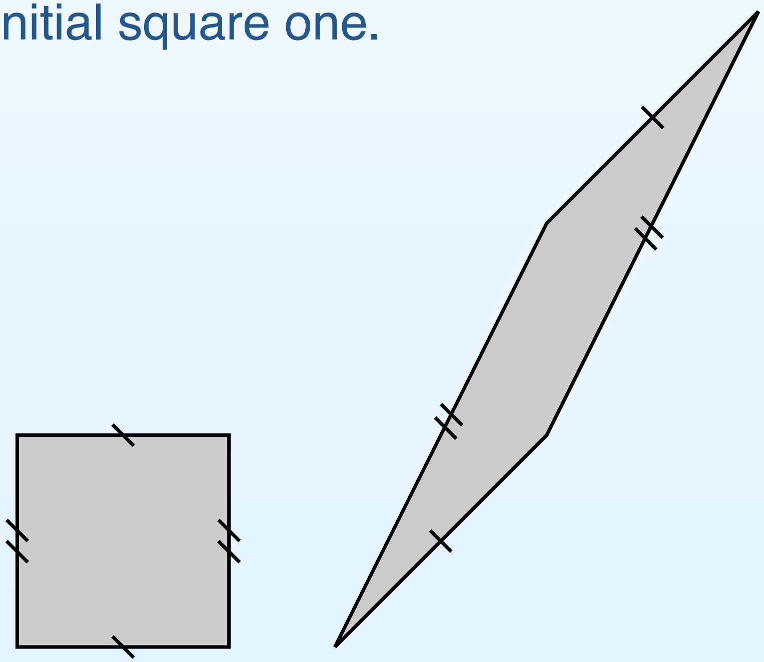
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

It corresponds to the integer linear map $\hat{g} : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Cutting and pasting appropriately the image parallelogram pattern we can check by hands that we can transform the new pattern to the initial square one.



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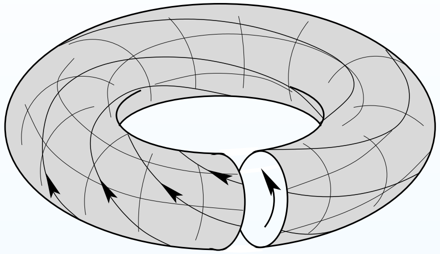
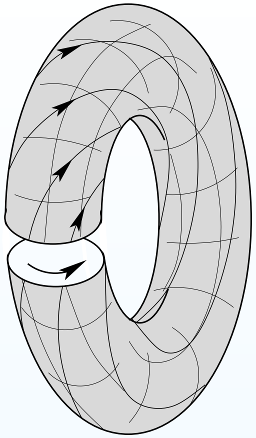
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

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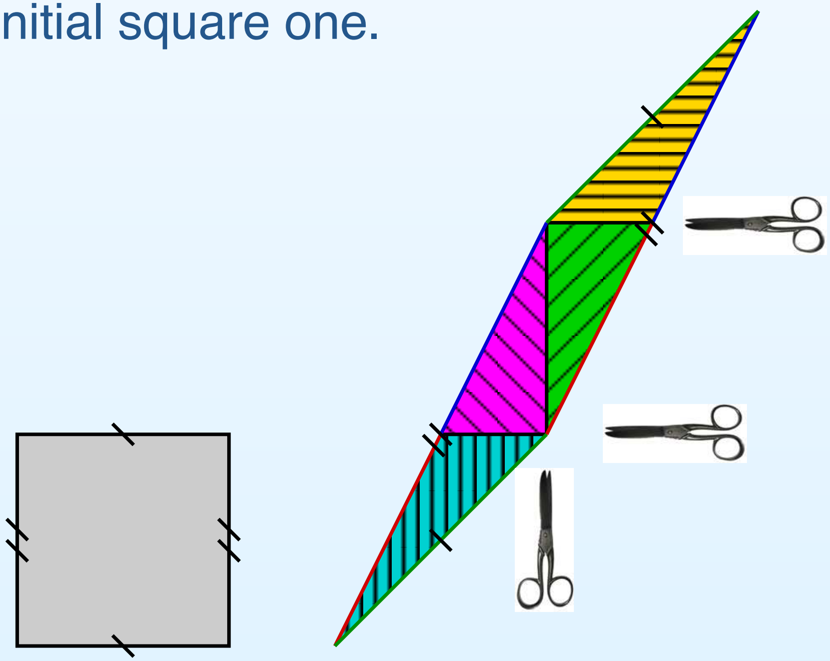

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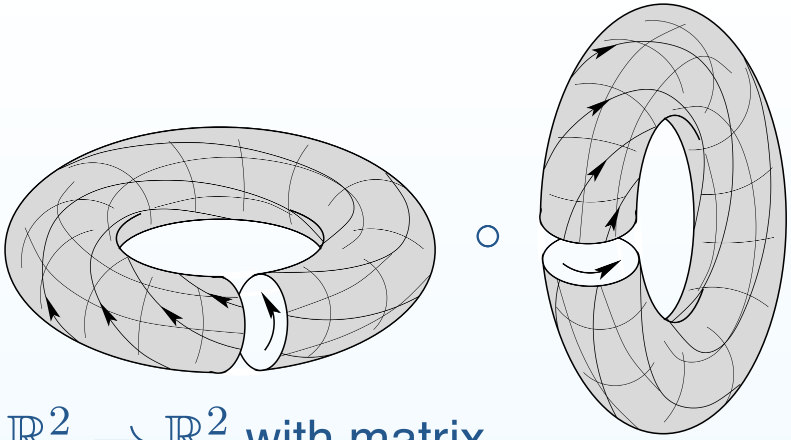
Consider a composition of two Dehn twists $g = f_v \circ f_h =$ 

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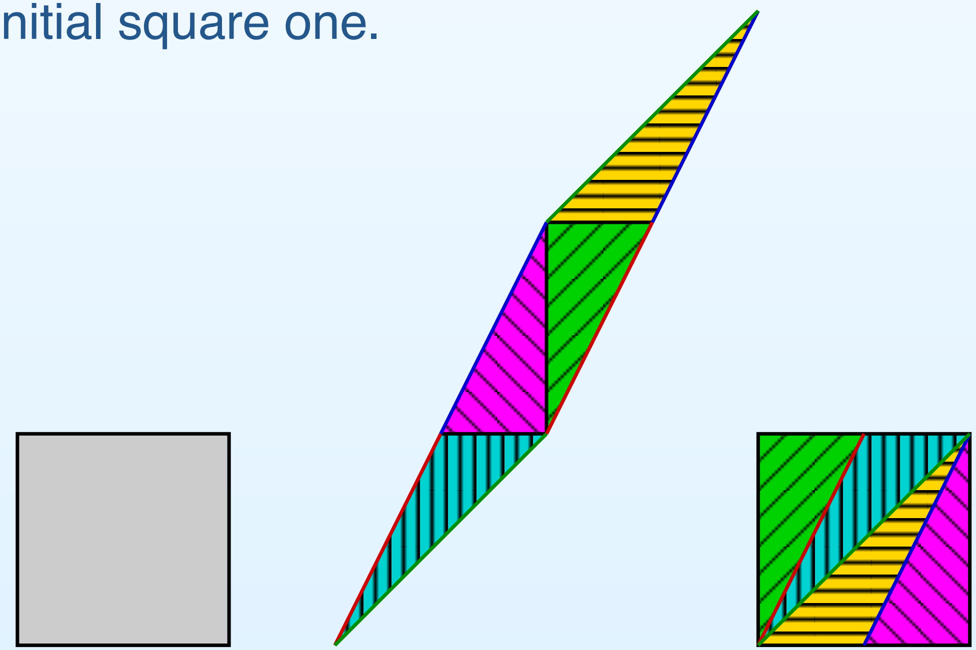

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Consider eigenvectors $\vec{v}_u$ and $\vec{v}_s$ of the linear transformation $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ with eigenvalues $\lambda = (3+\sqrt{5})/2 \approx 2.6$ and $1/\lambda = (3-\sqrt{5})/2 \approx 0.38$. Consider two transversal foliations on the original torus in directions $\vec{v}_u, \vec{v}_s$. We have just proved that expanding our torus $\mathbb{T}^2$ by factor λ in direction $\vec{v}_u$ and contracting it by the factor λ in direction $\vec{v}_s$ we get the original torus.

Definition. Surface automorphism homogeneously expanding in direction of one foliation and homogeneously contracting in direction of the transverse foliation is called a *pseudo-Anosov* diffeomorphism.

Consider a one-parameter family of flat tori obtained from the initial square torus by a continuous deformation expanding with a factor e^t in directions $\vec{v}_u$ and contracting with a factor e^t in direction $\vec{v}_s$. By construction such one-parameter family defines a closed curve in the space of flat tori: after the time $t_0 = \log \lambda_u$ it closes up and follows itself.

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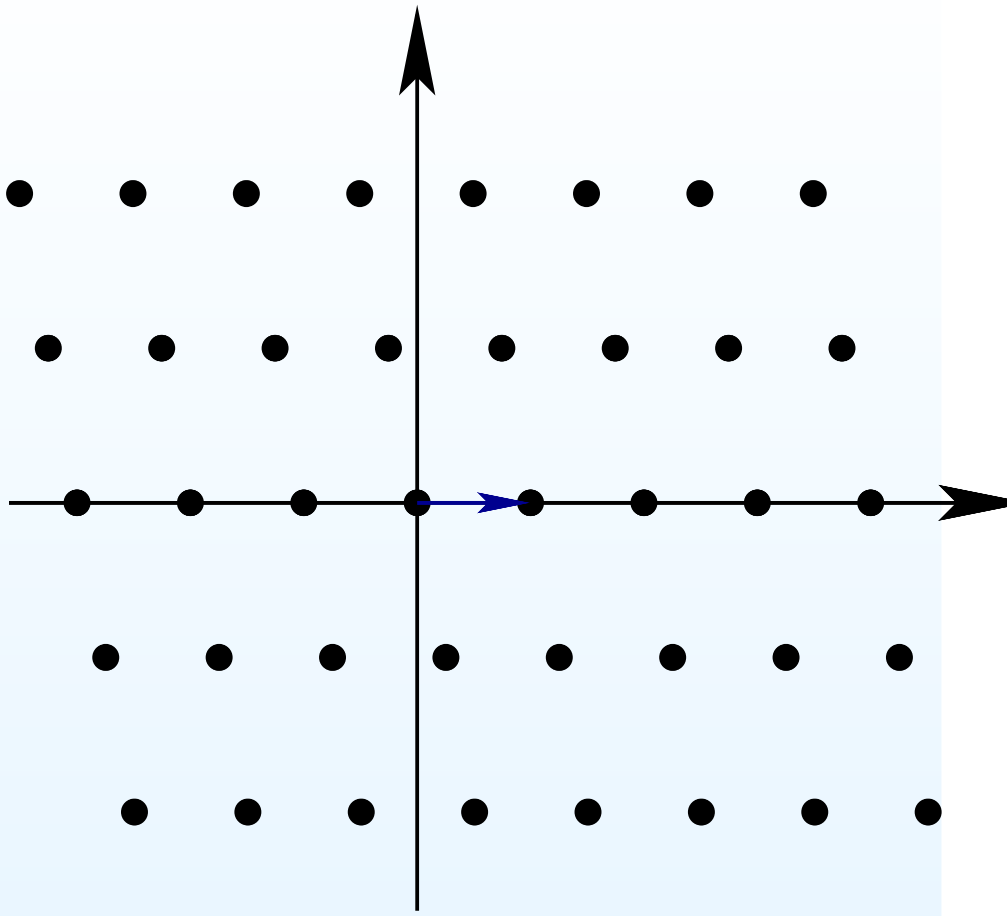
Space of lattices

- By a composition of homothety and rotation we can place the shortest vector of the lattice to the horizontal unit vector.



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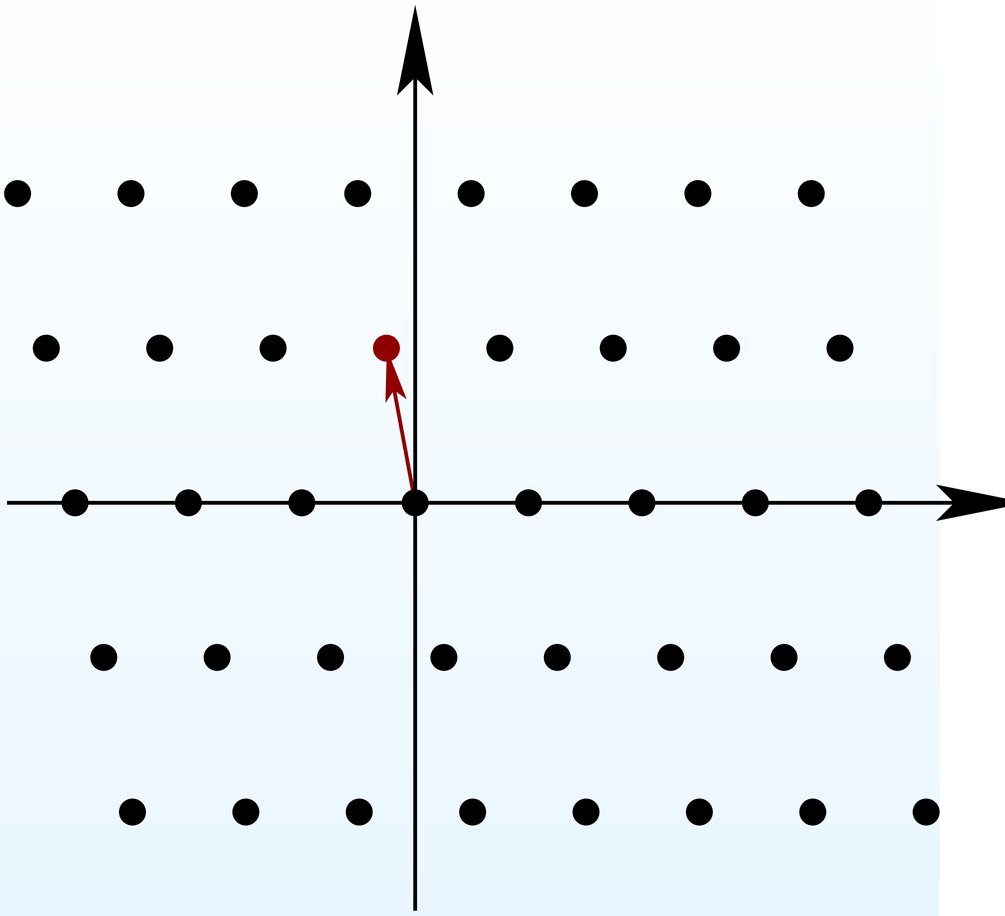

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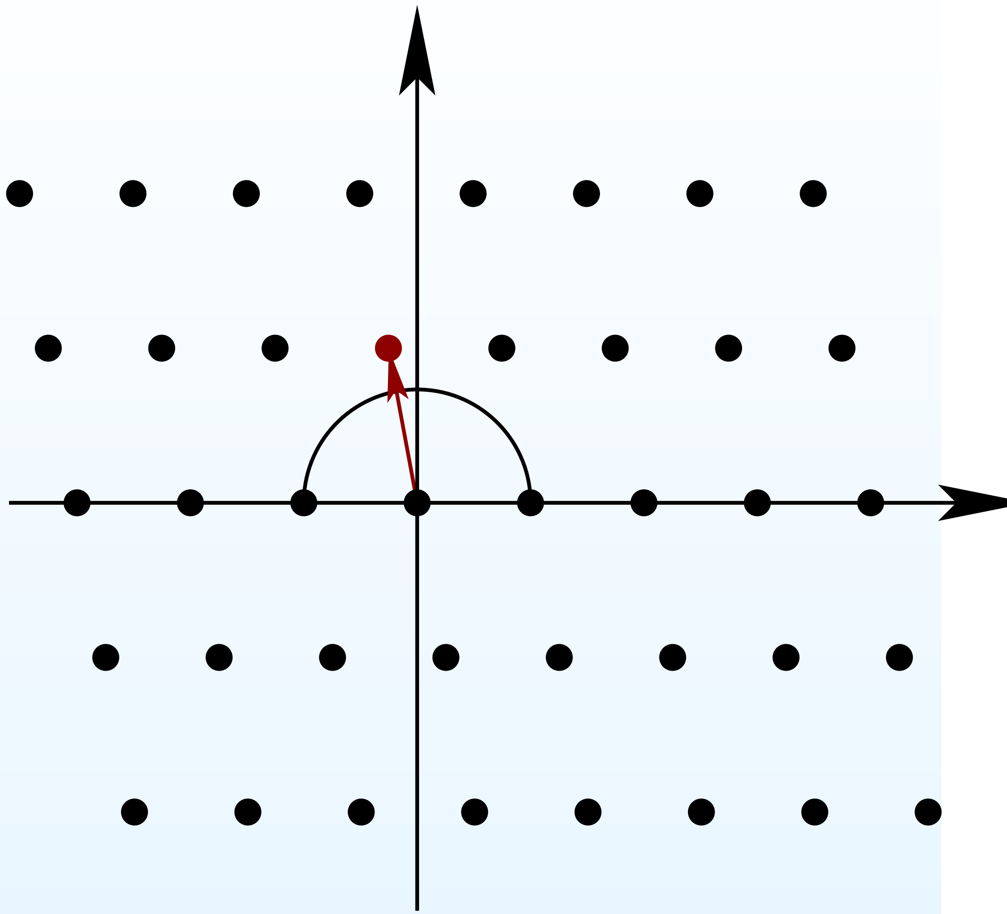

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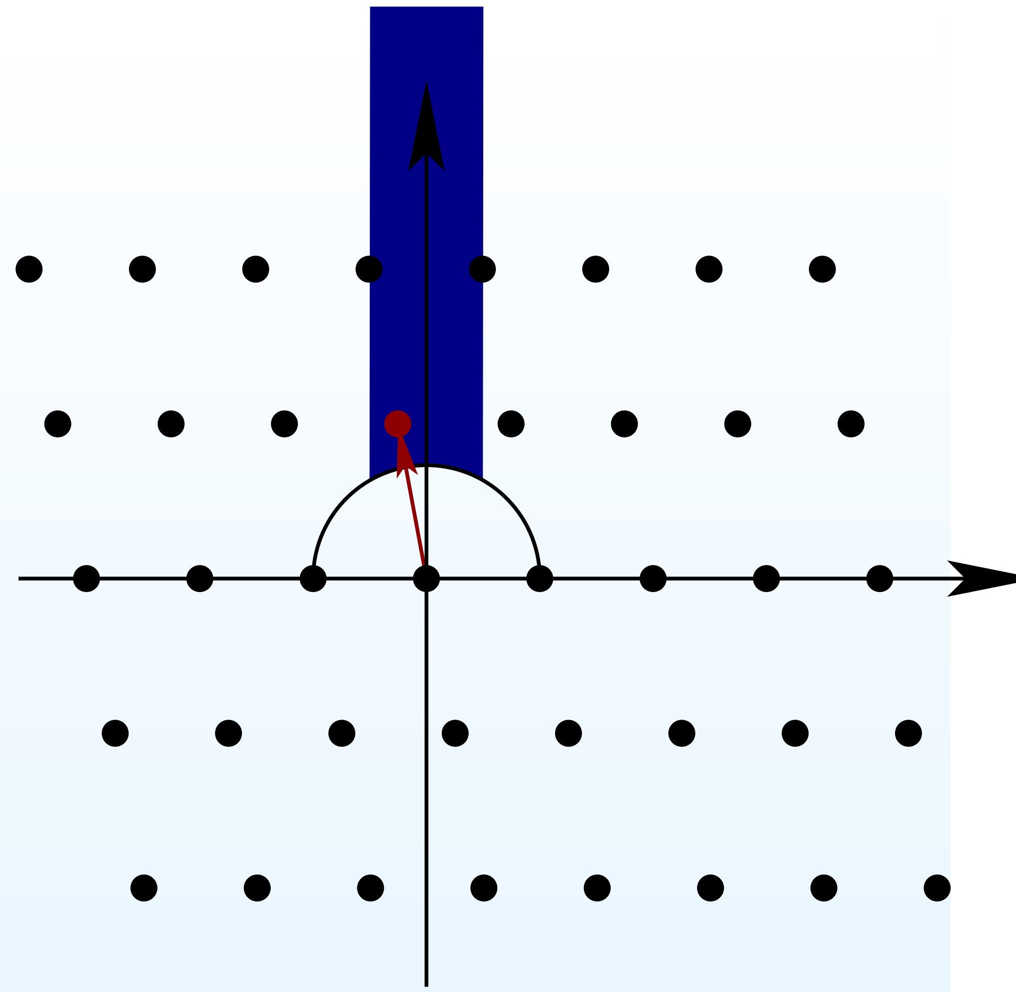

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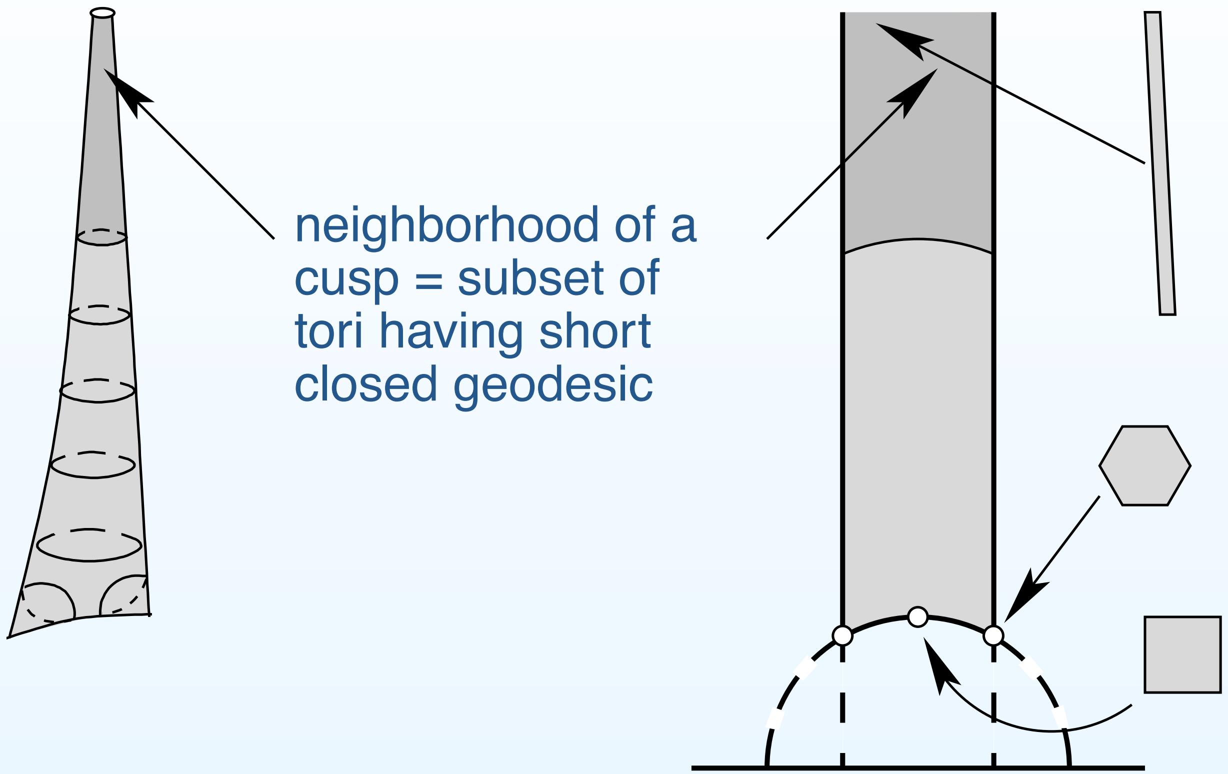
- Consider the lattice point closest to the origin and located in the upper half-plane.
- This point is located outside of the unit disc.
- It necessarily lives inside the strip $-1/2 \leq x \leq 1/2$.

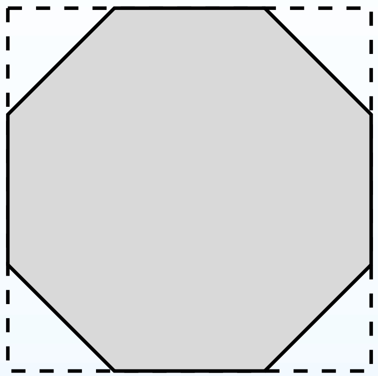
We get a fundamental domain in the space of lattices, or, in other words, in the moduli space of flat tori.

Moduli space of tori



The corresponding modular surface is not compact: flat tori representing points, which are close to the cusp, are almost degenerate: they have a very short closed geodesic. It also have orbifoldic points corresponding to tori with extra symmetries.

Very flat surface of genus 2



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

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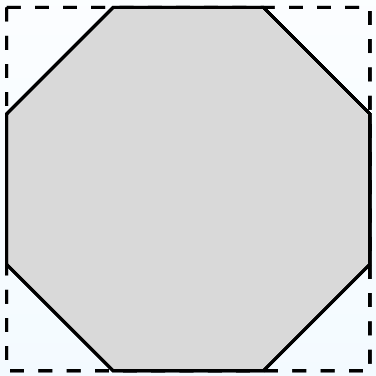

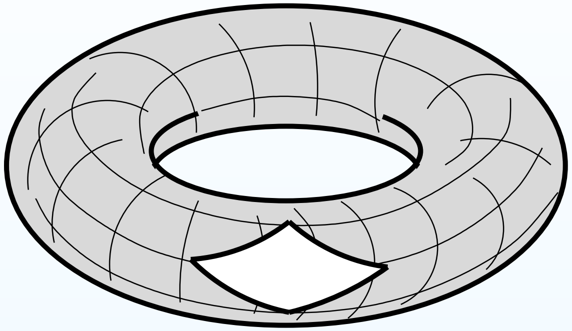

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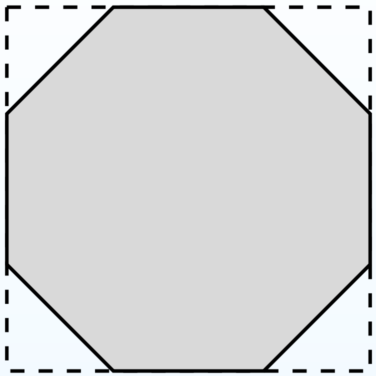

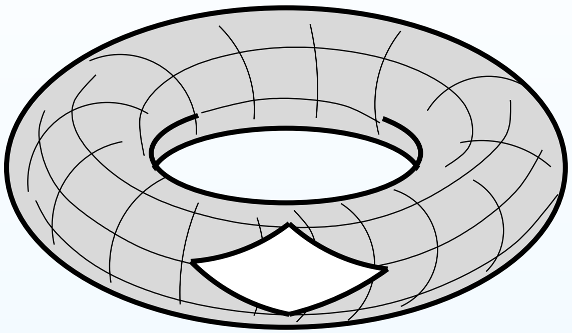

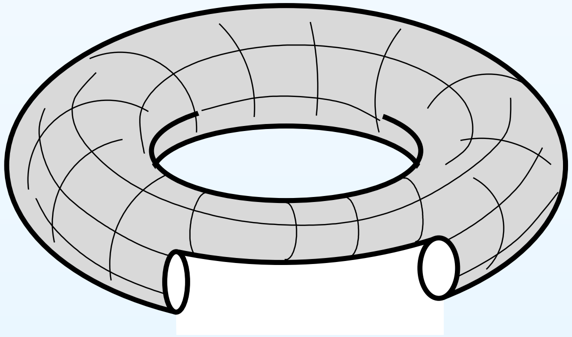

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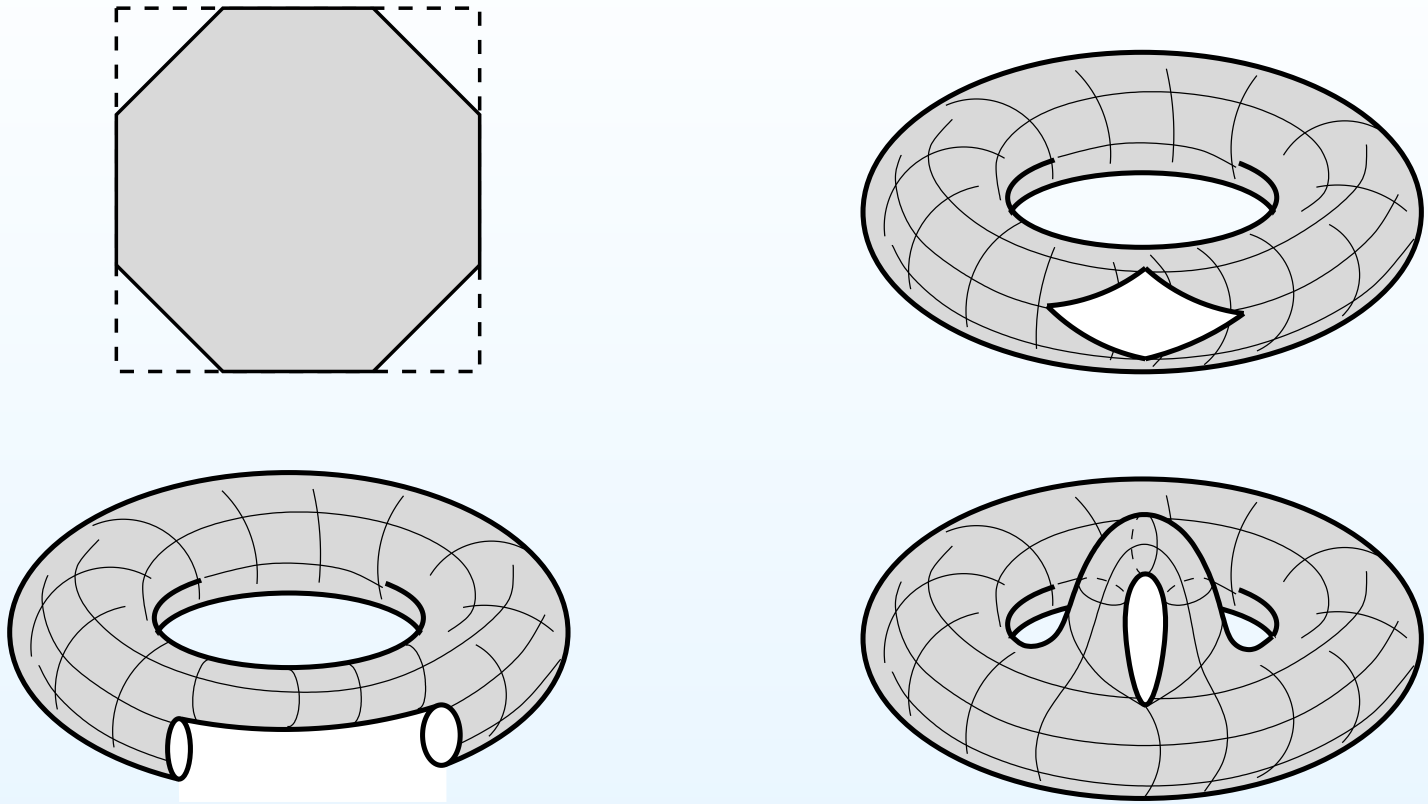

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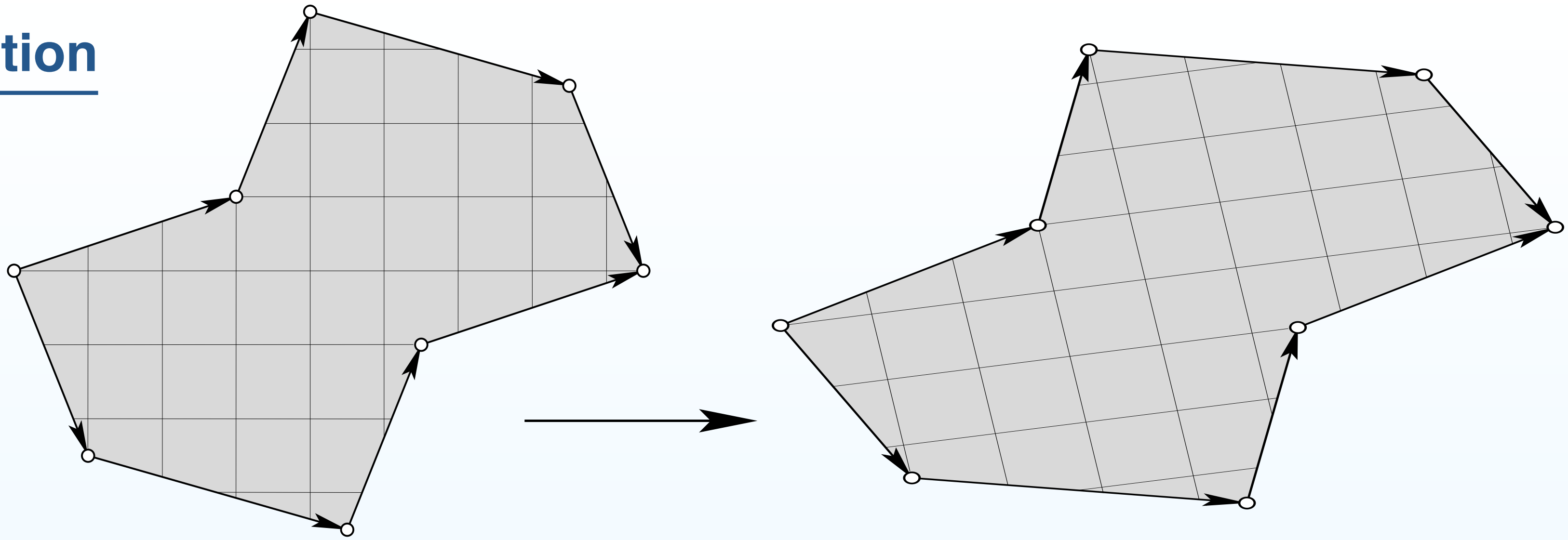
Group action



The group $\mathrm{SL}(2,\mathbb{R})$ acts on the each space $\mathcal{H}_1(d_1,\dots,d_n)$ of flat surfaces of unit area with conical singularities of prescribed cone angles $2\pi(d_i+1)$. This action preserves the natural measure on this space. The diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix} \subset \mathrm{SL}(2,\mathbb{R})$ induces a natural flow on $\mathcal{H}_1(d_1,\dots,d_n)$ called the *Teichmüller geodesic flow*.

Keystone Theorem (H. Masur; W. A. Veech, 1992). *The action of the groups $\mathrm{SL}(2,\mathbb{R})$ and $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}$ is ergodic with respect to the natural finite measure on each connected component of every space $\mathcal{H}_1(d_1,\dots,d_n)$.*

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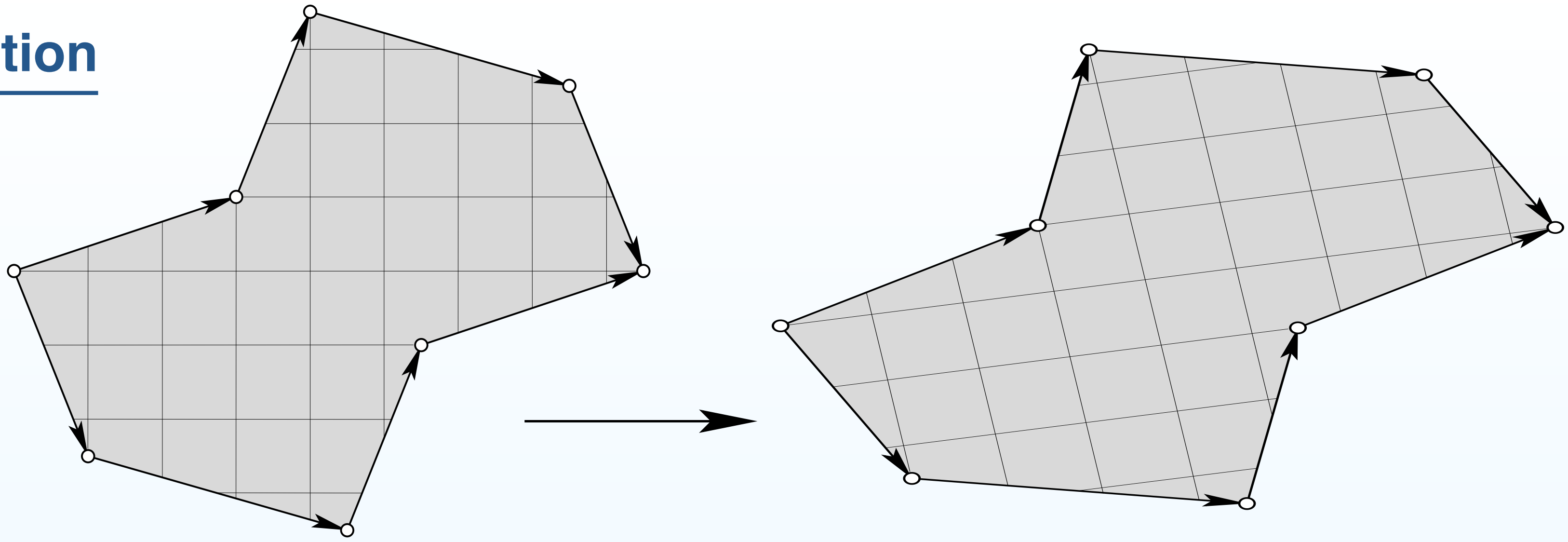

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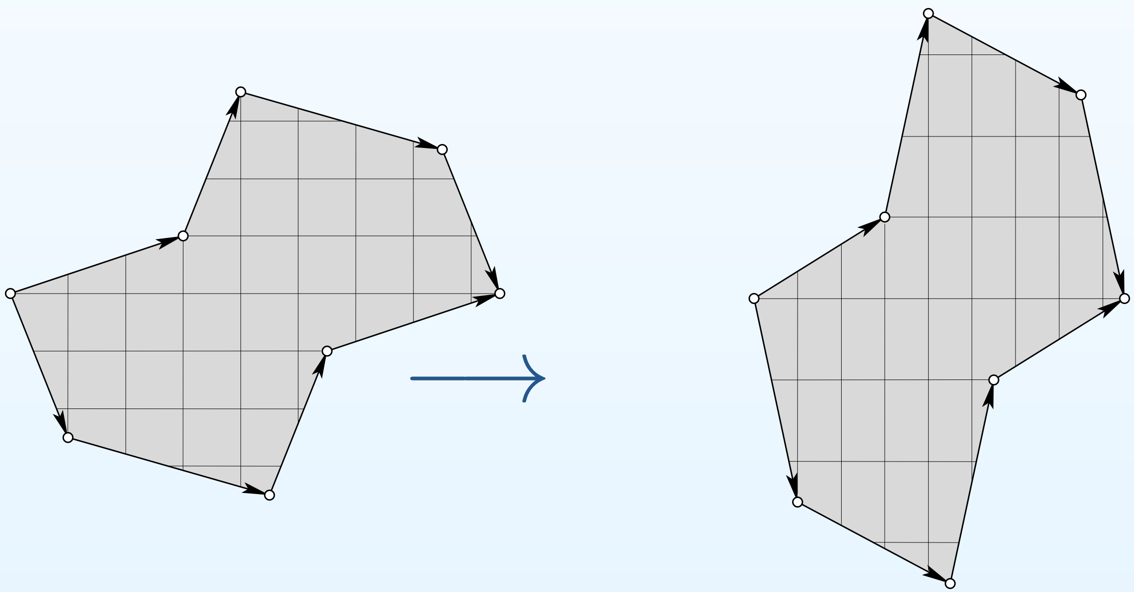

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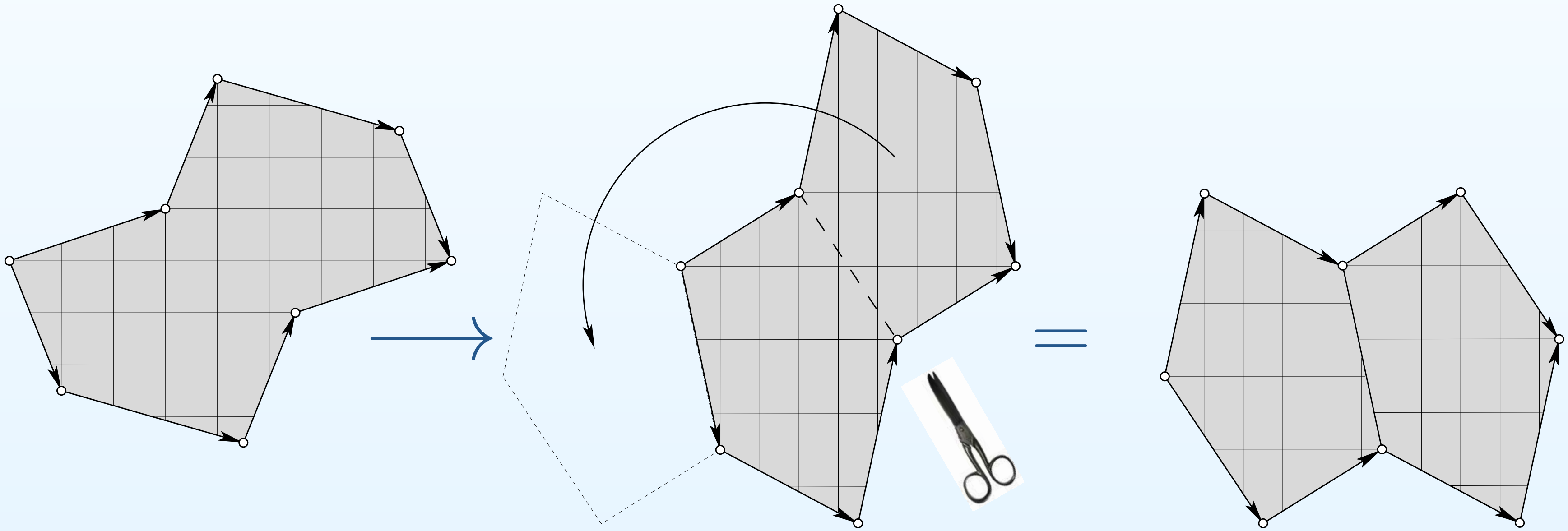
There is no paradox since we are allowed to cut-and-paste!



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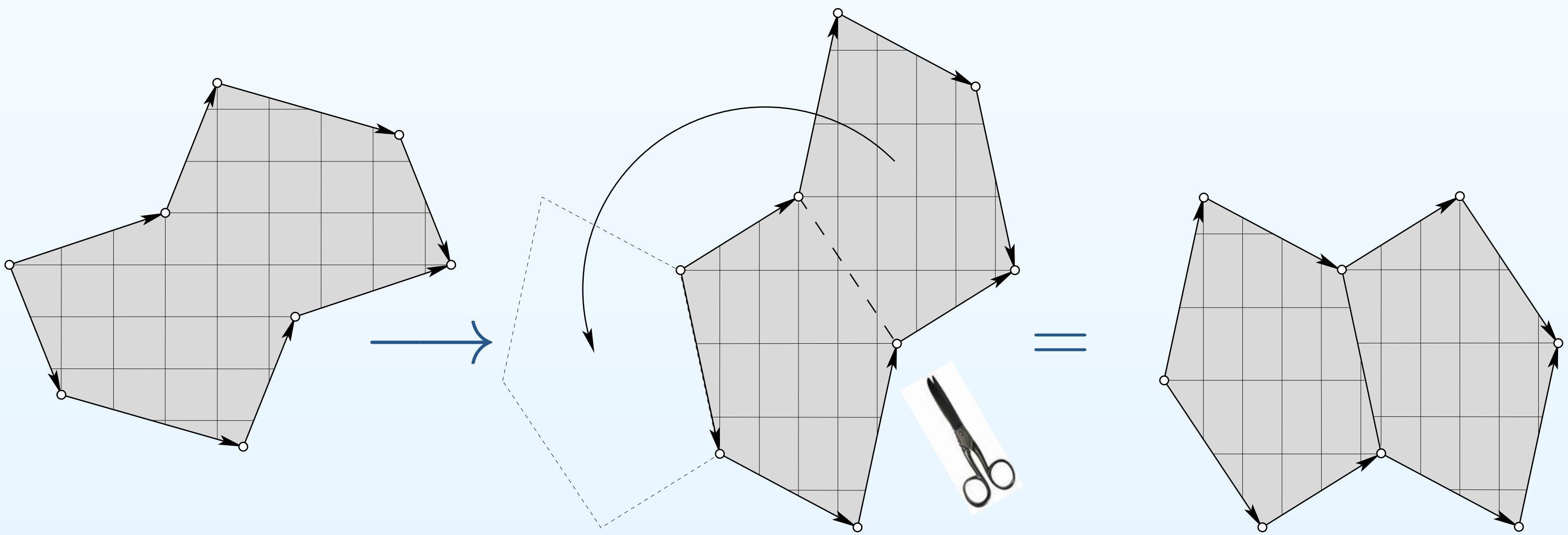
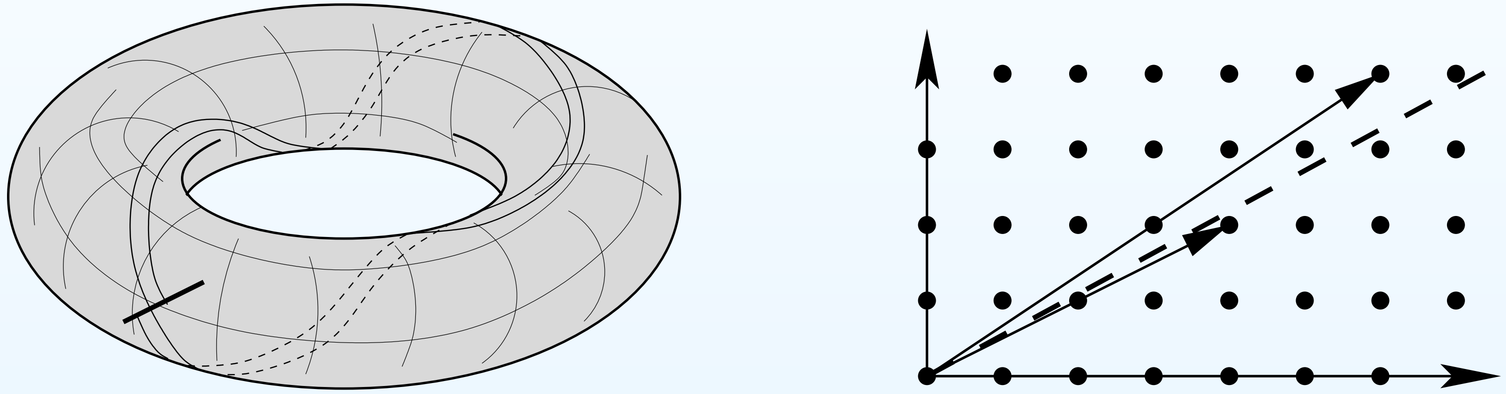


The first modification of the polygon changes the flat structure while the second one just changes the way in which we unwrap the flat surface.

2. Asymptotic flag of an orientable measured foliation

Asymptotic cycle for a torus

Consider a leaf of a measured foliation on a surface. Choose a short transversal segment X. Each time when the leaf crosses X we join the crossing point with the point x_0 along X obtaining a closed loop. Consecutive return points $x_1, x_2, \ldots$ define a sequence of cycles $c_1, c_2, \ldots$.



The *asymptotic cycle* is defined as $\lim_{n\to\infty} \dfrac{c_n}{n} = c \in H_1(\mathbb{T}^2; \mathbb{R})$.

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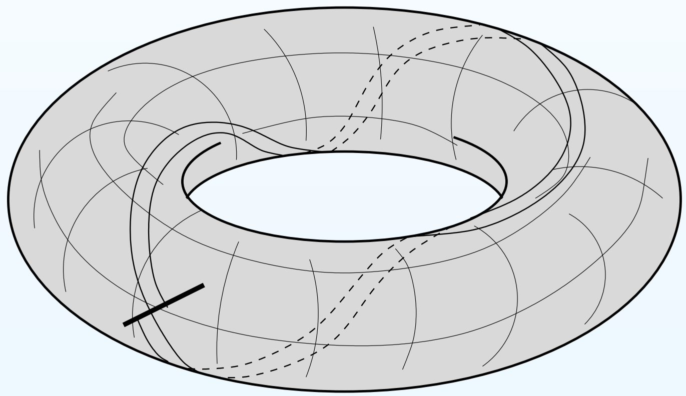

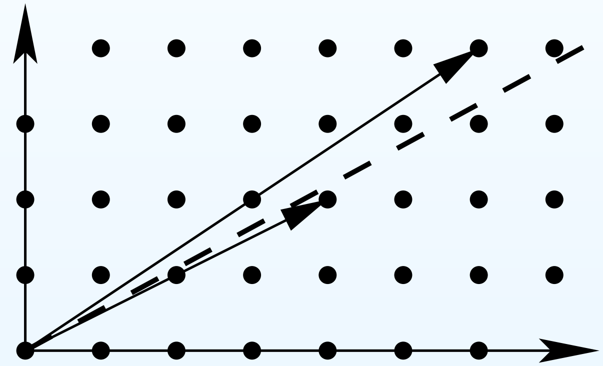

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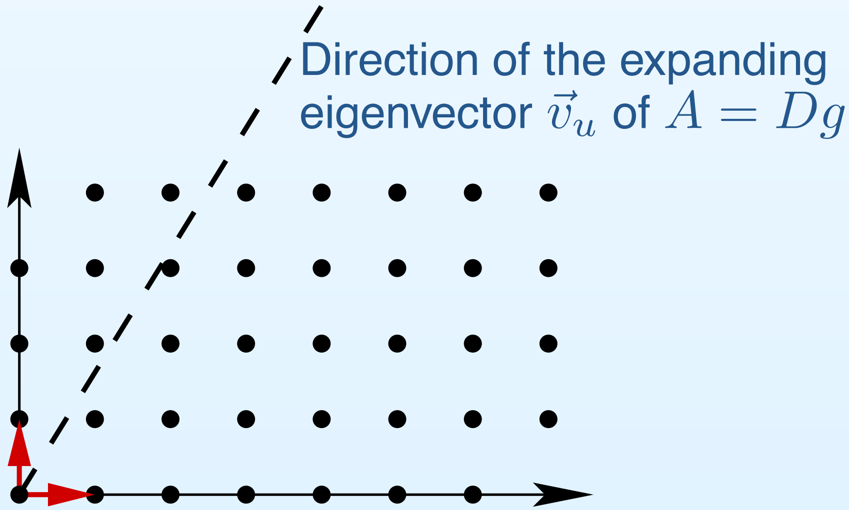
Asymptotic cycle in the pseudo-Anosov case

Consider a model case of the foliation in direction of the expanding eigenvector $\vec{v}_u$ of the Anosov map $g : \mathbb{T}^2 \to \mathbb{T}^2$ with $Dg = A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Take a closed curve γ and apply to it k iterations of g. The images $g_*^{(k)}(c)$ of the corresponding cycle $c = [\gamma]$ get almost collinear to the expanding eigenvector $\vec{v}_u$ of A, and the corresponding curve $g^{(k)}(\gamma)$ closely follows our foliation.

The first return cycles to a short subinterval exhibit exactly the same behavior by a simple reason that they are images of the first return cycles to a longer subinterval under a high iteration of g.



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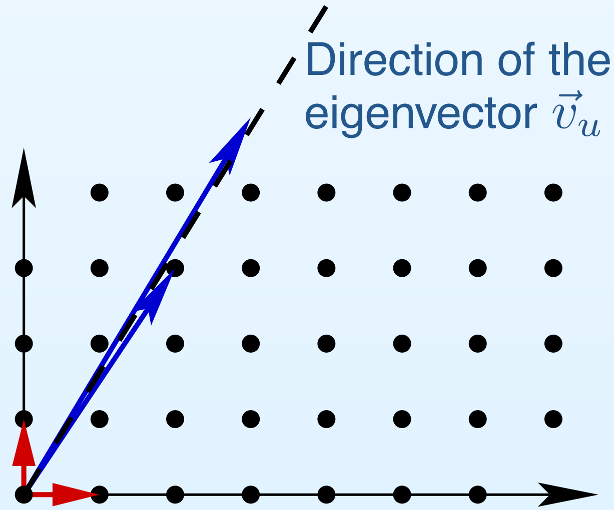

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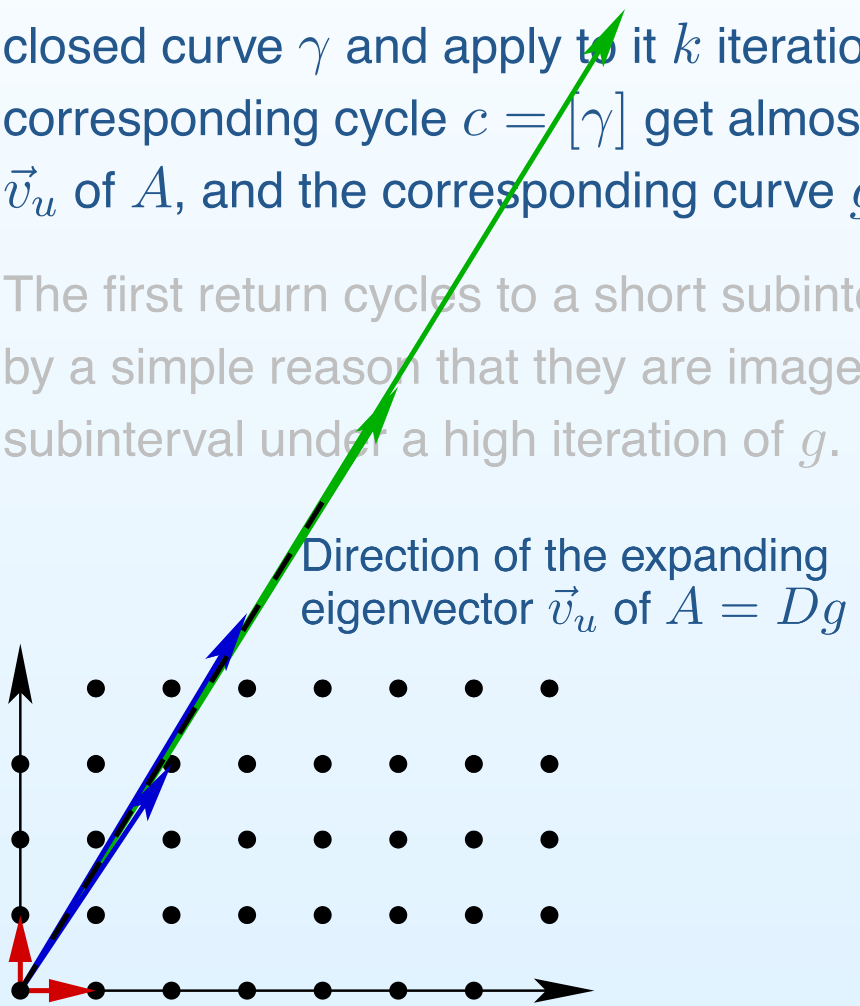

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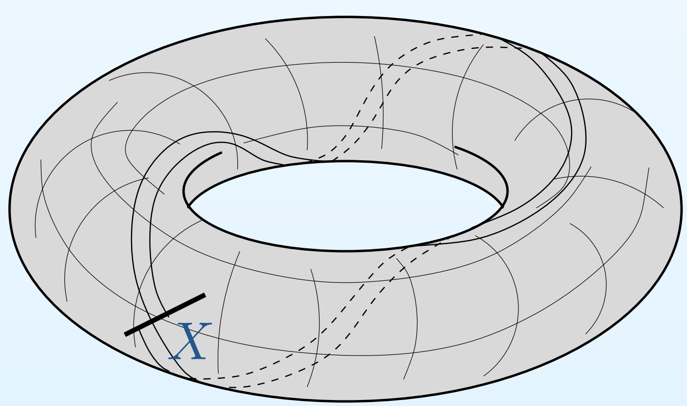
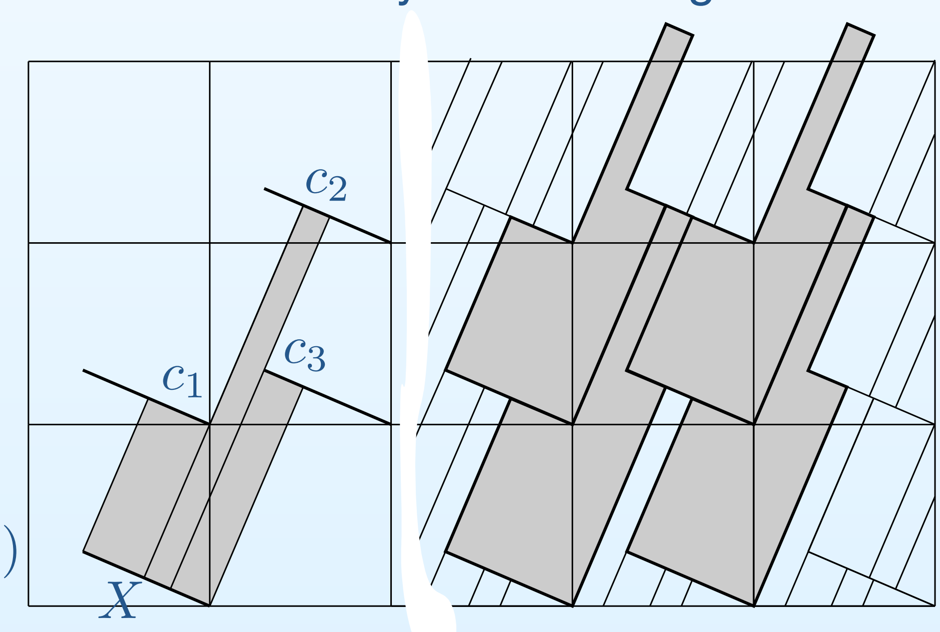
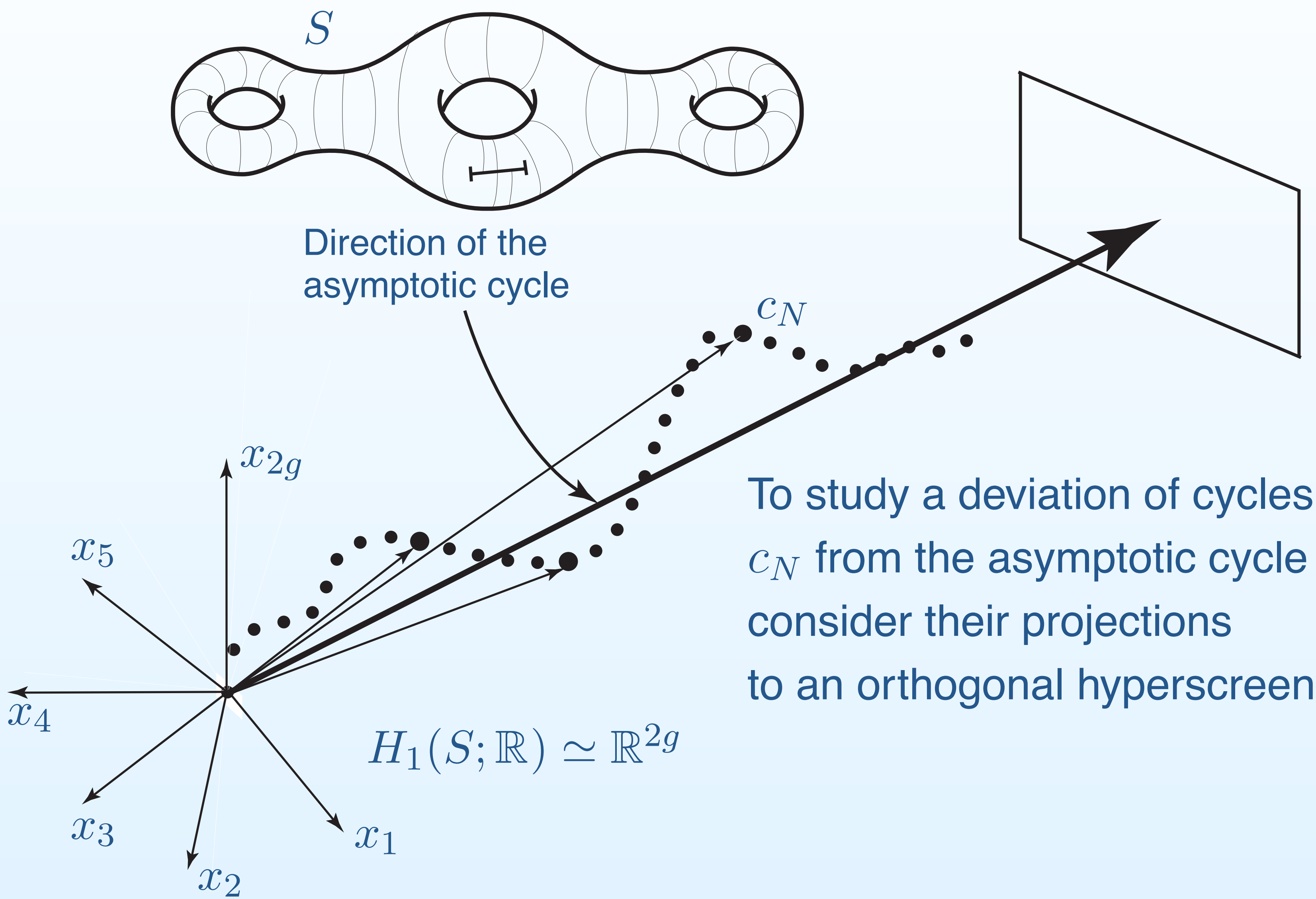




First return cycle $c_i(g(X))$ to $g(X)$ is $g_*(c_i(X))$

Asymptotic flag: empirical description



To study a deviation of cycles c_N from the asymptotic cycle consider their projections to an orthogonal hyperscreen

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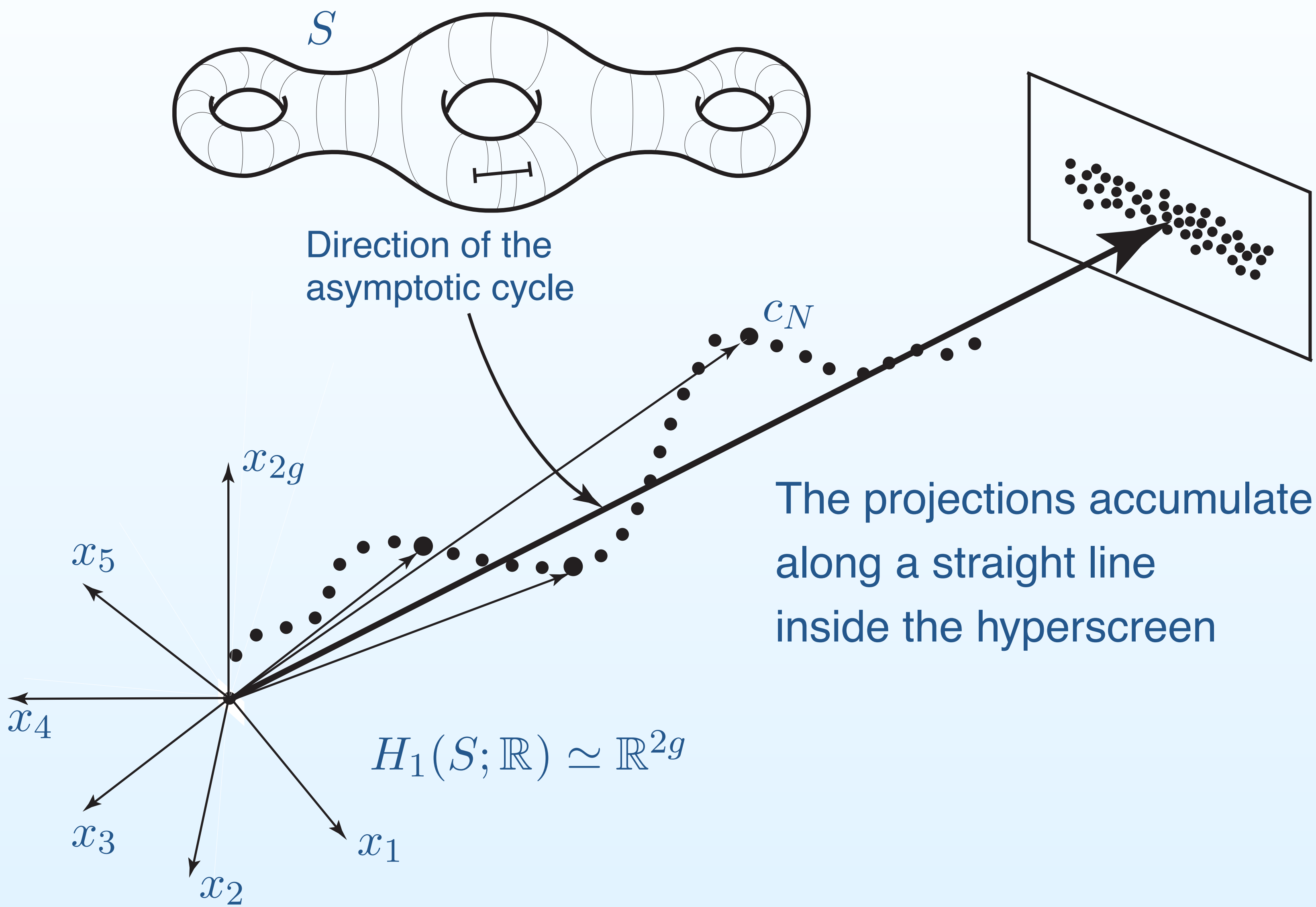

The projections accumulate along a straight line inside the hyperscreen

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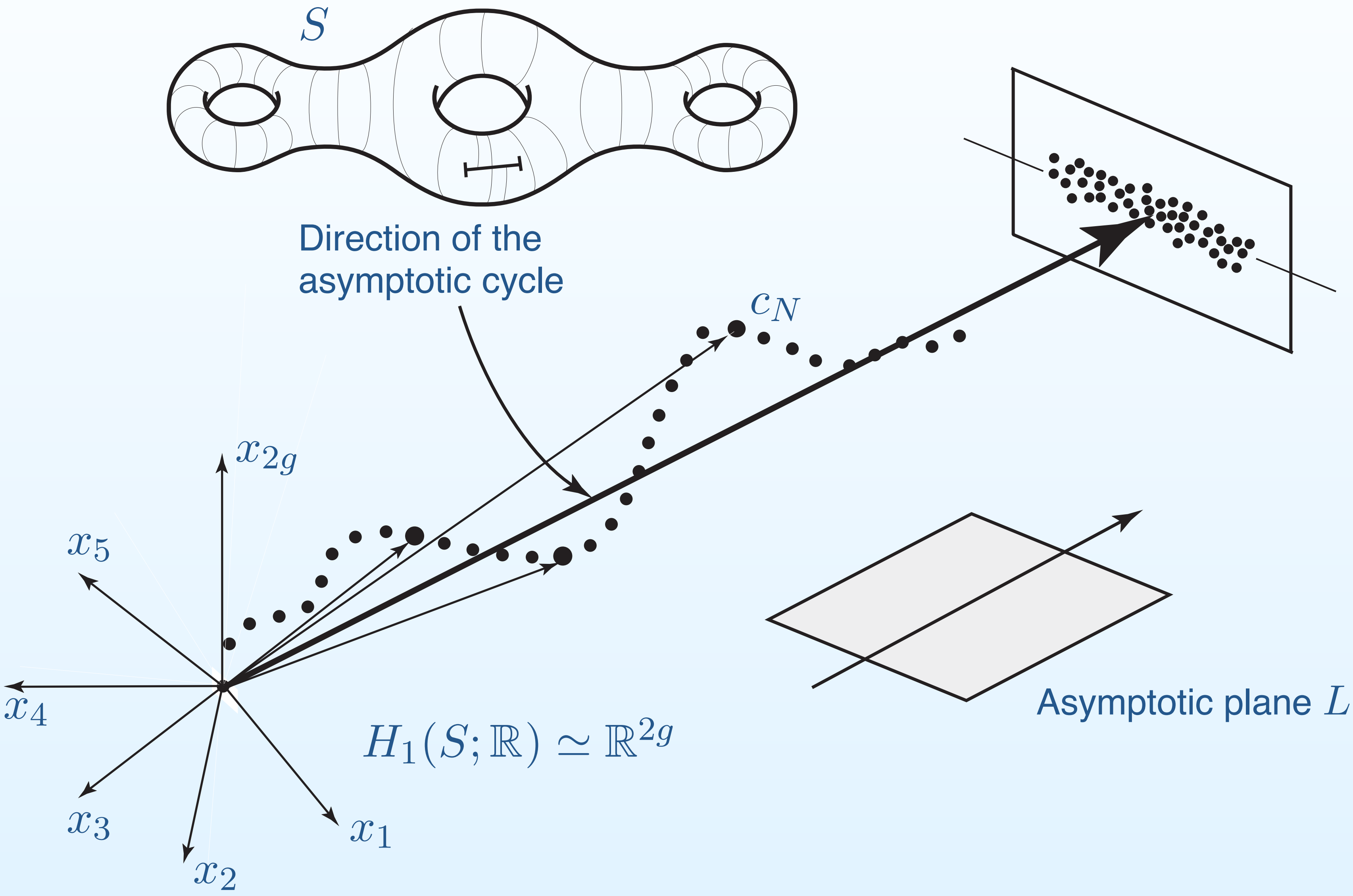

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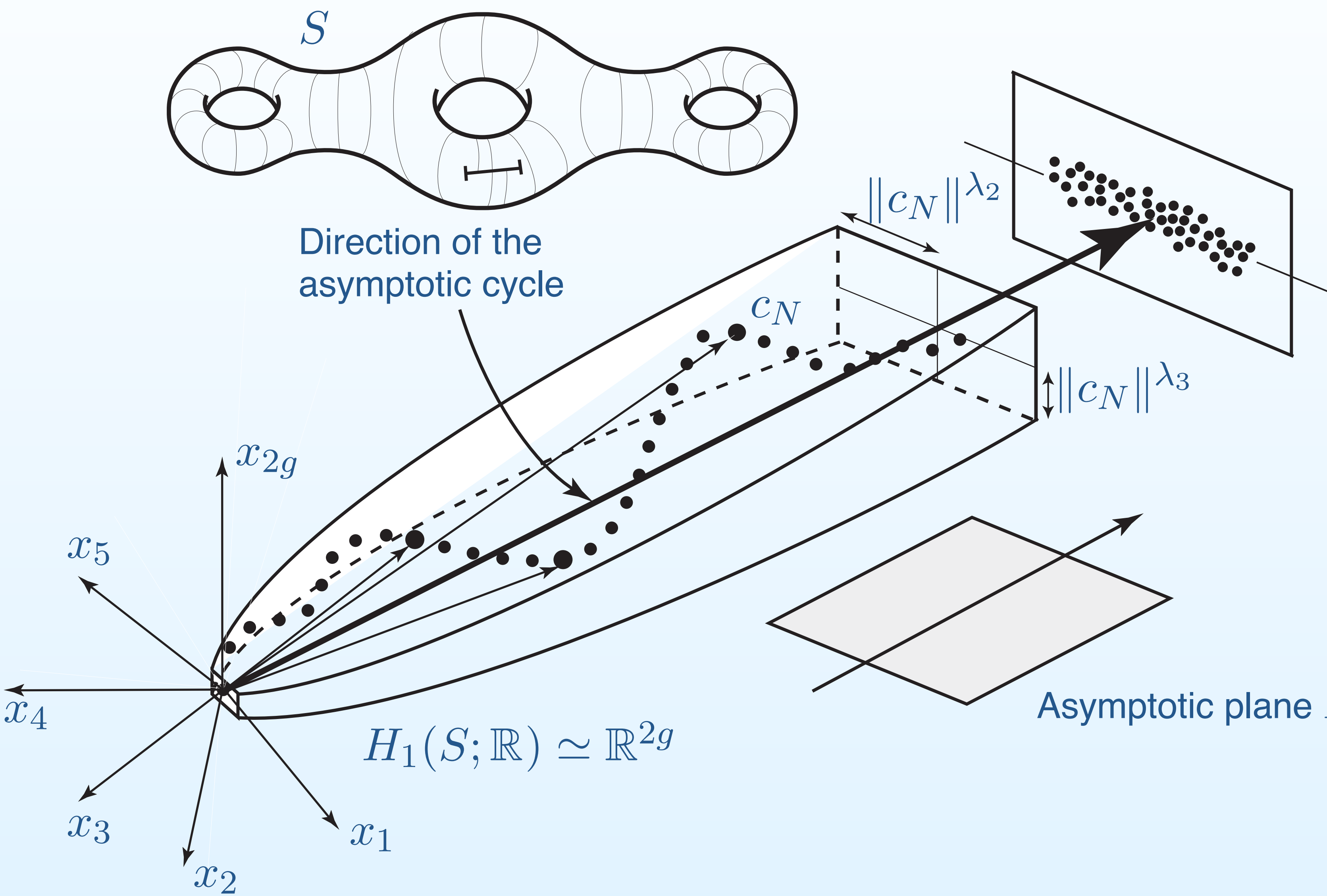

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Theorem (A. Z. , 1999) *For almost any surface* S *in any stratum* $\mathcal{H}_1(d_1,\dots,d_n)$ *there exists a flag of subspaces* $L_1 \subset L_2 \subset \cdots \subset L_g \subset H_1(S;\mathbb{R})$ *such that for any* $j = 1,\dots,g-1$

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and

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where the constant depends only on S *and on the choice of the Euclidean structure in the homology space.*

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The monodromy matrices of this cocycle are symplectic which implies that the Lyapunov exponents are symmetric:

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3. State of the art

Formula for the Lyapunov exponents

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$$\lambda_1 + \lambda_2 + \dots + \lambda_g \;=\; \frac{1}{12}\cdot\sum_{i=1}^{n}\frac{d_i(d_i+2)}{d_i+1} + \frac{\pi^2}{3}\cdot c_{area}(\mathcal{L})\,.$$

The proof is based on the initial Kontsevich formula $+$ analytic Riemann-Roch theorem $+$ analysis of $\det\Delta_{flat}$ under degeneration of the flat metric.

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Fantastic Theorem (A. Eskin, M. Mirzakhani, A. Mohammadi, 2014). *The closure of any* $\mathrm{SL}(2,\mathbb{R})$*-orbit is a suborbifold. In period coordinates any* $\mathrm{SL}(2,\mathbb{R})$*-suborbifold is represented by an affine subspace.*

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Developement (A. Wright, 2014) *Effective methods of construction of orbit closures.*

Theorem (J. Chaika, A. Eskin, 2014). *For any given flat surface* S *almost all vertical directions define a Lyapunov-generic point in the orbit closure of* $\mathrm{SL}(2,\mathbb{R})\cdot S$.

Solution of the generalized windtree problem (V. Delecroix–A. Z., 2015). Notice that any "windtree flat surface" S is a cover of a surface S_0 in the hyperelliptic locus $\mathcal{L}$ in genus 1, and that the cycles h and v are induced from S_0. Prove that the orbit closure of S_0 is $\mathcal{L}$. Using the volumes of the strata in genus zero, compute $c_{area}(\mathcal{L})$. Using the formula for $\sum \lambda_i$ compute λ_1.

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- Study extremal properties of the “curvature” of the Lyapunov subbundles compared to holomorphic subbundles of the Hodge bundle. Estimate the individual Lyapunov exponents.
- Prove conjectural formulae for asymptotics of volumes, and of Siegel–Veech constants when $g \to \infty$. (Partial results are already obtained by D. Chen–M. Möller–D. Zagier, 2015–, and by A. Eskin and A. Z. , 2015–)
- Express $c_{area}(\mathcal{L})$ in terms of an appropriate intersection theory (in the spirit of ELSV-formula for Hurwitz numbers or Mirzakhani formula for WP-volumes).
- Study dynamics of the Hodge bundle over other families of compact varieties (some experimental results for families of Calabi–Yau varieties are recently obtained by M. Kontsevich; some results for families of K3 surfaces were obtained by S. Filip). Are there other dynamical systems, which admit renormalization leading to dynamics on families of complex varieties?

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Billiard in a polygon: artistic image

Varvara Stepanova. Joueurs de billard. Thyssen Museum, Madrid