Statistical properties of the maximal entropy measure for partially hyperbolic attractors

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Entropy

Let $f : X \to X$ be a uniformly continuous map on a locally compact metric space. Given $\epsilon > 0$ and K compact let $S(n, \epsilon, K)$ denote the greatest cardinality of a (n, ϵ)-separable subset of K. The *topological entropy (following Bowen) of* f *relative to a (not necessarily invariant) compact set* K of X:

$$h(f, K) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \epsilon, K).$$

The *topological entropy* of $f : X \to X$, is defined by

$$h(f) := \sup \{ h(f, K); K \text{ compact } \}$$

Metric Entropy

Definition by Kolmogorov:

Let $(X, \mathcal{B}, \mu)$ be a measure space. The entropy of a finite partition $\mathcal{P}$ of X is:

$$h_\mu(\mathcal{P}) := -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

The entropy of a partition with respect to f is:

$$h_\mu(f, \mathcal{P}) := \lim_{n \to \infty} \frac{1}{n} h_\mu(\mathcal{P} \vee f^{-1}(\mathcal{P}) \vee \cdots \vee f^{n-1}(\mathcal{P})).$$

The metric entropy of f with respect to μ is given by

$$h_\mu(f) := \sup \{ h_\mu(f, \mathcal{P}); \mathcal{P} \text{ finite } \}.$$

Variational principle

By the **variational principle**

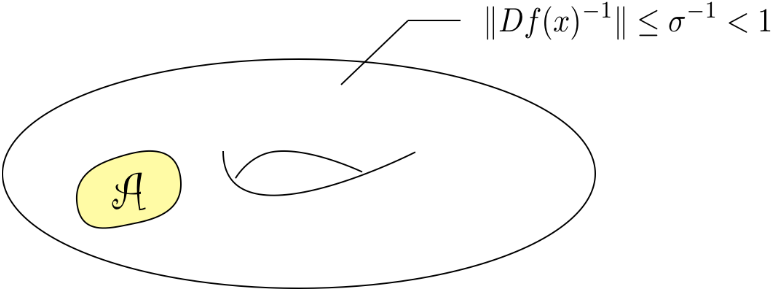
$$h(f) = \sup \left\{ h_\mu(f); \mu \in \mathcal{M}_f^1(X) \right\}$$

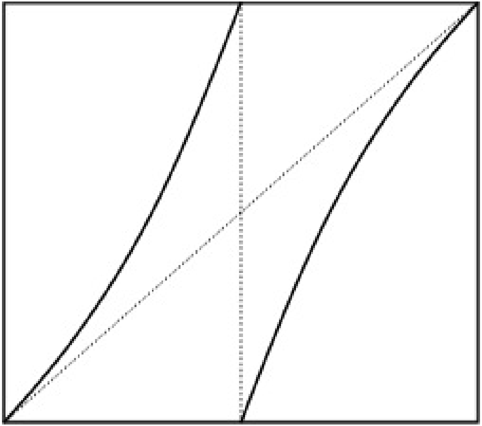
An f-invariant probability measure μ is the maximal entropy measure for f if it realizes the supremmum, that is,

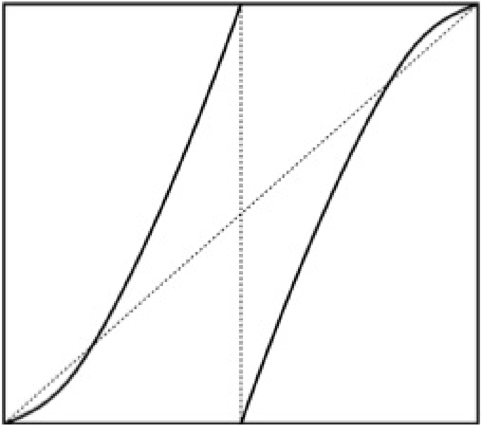
$$h(f) = h_\mu(f).$$

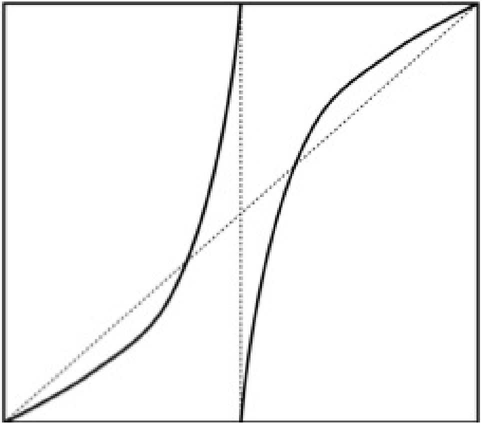
Some very recent results

- Buzzi, Fisher, Sambarino, Vásquez: Maximal entropy measures for certain partially hyperbolic, derived from Anosov systems (2012).
- Climenhaga, Fisher, Thompson: Unique Equilibrium States for the robustly transitive diffeomorphisms of Mañé and Bonatti-Viana (2015)
- A. Castro, P. Varandas: Equilibrium states for non-uniformly expanding maps: decay of correlations and strong stability (2013).


$\|Df(x)^{-1}\| \leq \sigma^{-1} < 1$
$\mathcal{A}$







Other recent results on Statistical Properties for Equilibrium States

- V. Baladi, Anisotropic Sobolev spaces and dynamical transfer operators: C^∞ foliations, (2005)
- V. Baladi and S. S. Gouezel. Banach spaces for piecewise cone hyperbolic maps, (2010).
- V. Baladi and C. Liverani, Exponential decay of correlations for piecewise cone hyperbolic contact flows, (2012).
- V. Baladi and M. Tsujii, Hölder and Sobolev spaces for hyperbolic diffeomorphisms., (2007).
- S. Gouezel and C. Liverani. Banach spaces adapted to Anosov systems, (2006).

First Setting

- $f : M \longrightarrow M$: diffeomorphism partially hyperbolic diffeomorphism on a compact manifold M.
- f is semiconjugated to a map $g : N \longrightarrow N$ as in Castro-Varandas. That is, $\Pi \circ f = g \circ \Pi$,

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \Pi \downarrow & & \downarrow \Pi \\ N & \xrightarrow{g} & N \end{array} \tag{1}$$

where $\Pi : M \longrightarrow N$ is surjective and continuous.

First Setting

Aditional hypothesis $M = \dot{\bigcup}_{y \in N} M_y$ where $M_y := \Pi^{-1}(y)$.

Furthermore, $f : M_y \to M_{g(y)}$ is a λ_s-contraction. that is, there exists $0 < \lambda_s < 1$ such that

$$d(f(z), f(w)) \leq \lambda_s d(z, w)$$

for all $z, w \in M_y$.

Examples

A family of examples can be obtained from the skew-product

$$\begin{aligned} f: \quad N \times K &\to N \times K \\ (x, y) &\mapsto (g(x), \Phi(x, y)) \end{aligned}$$

where $g : N \to N$ is a non-uniformly expanding map as in [CV13] and $\Phi : N \times K \to K$ is a λ_s-contraction such that f be a diffeomorphism onto its image.

Derived from Solenoid

Example in the solid torus $S^1 \times D$:

$$f: \begin{array}{ll} S^1 \times D & \to S^1 \times D \\ (\theta, z) & \mapsto (g(\theta), \varphi(\theta) + A(z)) \end{array}$$

where g is the Manneville-Pomeau map given by

$$f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{, se } 0 \leq x \leq \frac{1}{2} \\ (x-1)(1 + 2^\alpha (1-x)^\alpha) + 1 & \text{, se } \frac{1}{2} < x \leq 1 \end{cases}$$

$\alpha \in (0,1)$, φ a local diffeomorphism A a linear contraction such that f be a diffeomorphism.

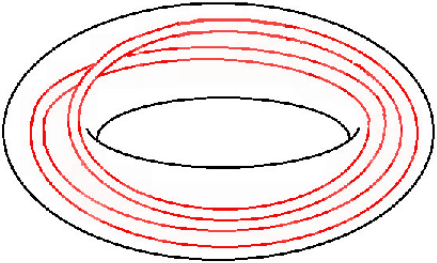


Figura: Solenoid

Theorem A

Theorem

(Existence and uniqueness of the Maximal entropy measure)
Let f a diffeomorphism as above. Then, there exists a unique maximal entropy measure μ for f.

Statistical Stability

Corollary

(Statistical Stability in the Derived from Solenoid case.) *Let f_n be a sequence of derived from solenoid diffeomorphisms and call μ_n the maximal entropy probability measure for f_n. If $f_n \longrightarrow f$ in the C^1-topology, then μ_n converges to the maximal entropy probability measure for f in the weak-* topology.*

Theorem B

Theorem

(Exponential Decay of Correlations) *The maximal entropy measure* μ *we constructed for* $f : \Lambda \to \Lambda$, *has exponential decay of correlation in the space of Hölder continuous observables, that is, there exists* $0 < \tau < 1$ *such that for all* $\varphi, \psi \in C^{\alpha}(M)$ *there exists* $K(\varphi, \psi) > 0$ *satisfying*

$$\left| \int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq K(\varphi, \psi) \cdot \tau^n, \text{ para todo } n \geq 1.$$

Theorem C

Theorem

(Central Limit Theorem)

Given φ a Hölder continous function

$$\sigma_\varphi^2 := \int \phi^2 d\mu + 2\sum_{j=1}^{\infty} \int \phi \cdot (\phi \circ f^j)\, d\mu, \text{ where } \phi = \varphi - \int \varphi\, d\mu.$$

Then $\sigma_\varphi < \infty$ and $\sigma_\varphi = 0$ if, and only if, $\varphi = u \circ f - u$ for some $u \in L^2(\mu)$. Furthermore, if $\sigma_\varphi > 0$ then, for all interval $A \subset \mathbb{R}$

$$\lim_{n\to\infty} \mu\left(A_\varphi^n\right) = \frac{1}{\sigma_\varphi\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma_\varphi^2}}\, dt.$$

Second Setting

Let M be a compact Riemannian manifold and $f : M \to M$ be a C^{1+}diffeomorphism. Assume that there exists a compact subset Λ of M with the following properties:

- There exists an open $f-$invariant neighborhood Q of Λ, such that $f(\overline{Q}) \subset Q$ and
$$\Lambda = \bigcap_{n=0}^{\infty} f^n(Q).$$
- Λ is partially hyperbolic, in the sense that there exists a Df-invariant dominated splitting
$$T_\Lambda M = E^{ss} \oplus E^{uc}, dim(E^{ss}) > 0$$
of the tangent bundle restricted to Λ, such that, once fixed a Riemannian metrics in M we have:

Second Setting

1. E^{ss} contracts uniformly: $\|Df^n|E^{ss}_x\| \leq C\lambda_s^n$
2. E^{uc} is dominated by E^{ss}: $\|Df^n|E^{ss}_x\|\|Df^{-n}|E^{uc}_{f^n(x)}\| \leq C\lambda_s^n$

for all $n \geq 1$ and $x \in \Lambda$, with $0 < \lambda_s < 1$.

Second Setting

- There exists an f-invariant center-unstable foliation $\mathcal{F}^{uc}_{loc}$ of a neighborhood Λ, which is tangent to the center unstable subbundle E^{uc} in Λ. There is also an $f-$invariant stable foliation $\mathcal{F}^{s}_{loc}$ tangent to the stable subbundle E^{ss} in Λ.

Second Setting

- f restricted to Λ admits a Markov partition $\mathcal{R} = \{R_1, \cdots, R_p\}$, $p \geq 2$ with the (mild) mixing property: given $i, j \in \{1, \cdots, p\}$, there exists $n_0 \geq 1$ such that

$$f^n(R_i) \cap R_j \neq \emptyset, \forall n \geq n_0.$$

Second Setting

We distinguish two kinds of rectangles in $\mathcal{R}$ according to its behavior in the direction E^{uc}. Fixed $0 < \zeta < 1$, we say $R_i \in \mathcal{R}$ is a *good rectangle* if

$$\|Df|_{E_x^{uc}}\|^{-1} \leq \zeta$$

for all $x \in R_i$. That is, E^{uc} expands uniformly in R_i, for one iterate. The other rectangles will be called bad rectangles.

- There exists at least one good rectangle and for all x in a bad rectangle

  $$\|Df|_{E_x^{uc}}\|^{-1} \leq L$$

  for some $L \geq 1$ close to 1 (depending on ζ and the combinatorics of the partition).

Existence of the measure

The semiconjugation between f and g guarantees that

$$h(f) \geq h(g).$$

Due to Bowen,

$$h(f) \leq h(g) + \sup\{h(f, \Pi^{-1}(y)); y \in N\}$$

By the λ_s-contraction of $f : M_y \to M_{g(y)}$ it follows that

$$h(f, \Pi^{-1}(y)) = 0$$

Hence $h(f) = h(g)$.

Existence of the measure

Let Π_Λ be the restriction of Π to the set Λ and $\mathcal{A}_N$ the Borel σ-algebra of N. Defining $\mathcal{A}_0 := \Pi_\Lambda^{-1}(\mathcal{A}_N)$ e $\mathcal{A}_n := f^n(\mathcal{A}_0)$ we obtain a sequence of σ-álgebras in Λ such that

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \cdots$$

Let $\mu_n : \mathcal{A}_n \to [0,1]$ defined by:

$$\mu_n(f^n(A_0)) = \nu(\Pi_\Lambda(A_0))$$

for all $A_0 \in \mathcal{A}_0$. Let $\mathcal{A} := \bigcup_{n=0}^{\infty} \mathcal{A}_n$, define $\mu : \mathcal{A} \to [0,1]$, $\forall A \in \mathcal{A}_n$ by

$$\mu(A) = \mu_n(A).$$

Existence and uniqueness of μ

By Brin-Katok we proved that

$$h_\mu(f) \geq h_\nu(g) = h(g) = h(f).$$

Suppose that μ_1 is another maximal entropy measure μ. Let

$$\nu_1 := \Pi_\Lambda * \mu_1$$

the push-forward of μ_1. Then ν_1 is also maximal entropy measure for g, which is a contradiction.

Projetive Metrics and Cones

Given a projective cone C, let

$$\alpha(v,w) := \sup\{t > 0; w - t \cdot v \in C\}$$

and

$$\beta(v,w) := \inf\{s > 0; s \cdot v - w \in C\}.$$

The projective metrics associated to the projective metrics C is given by:

$$\theta(v,w) := \log \frac{\beta(v,w)}{\alpha(v,w)}.$$

conventioning that $\theta = +\infty$ if $\alpha = 0$ or $\beta = +\infty$.

Projective Metrics and Cones

Theorem

Let E_1 and E_2 be vector spaces and let $C_1 \subset E_1$ e $C_2 \subset E_2$ be projective cones. If $L : E_1 \longrightarrow E_2$ is a linear operator such that $L(C_1) \subset C_2$ and $D = \sup \{\theta_2(L(v), L(w)); v, w \in C_1\} < \infty$ then

$$\theta_2(L(v), L(w)) \leq \left(1 - e^{-D}\right) \theta_1(v, w),$$

for all $v, w \in C_1$.

Ruelle-Perron-Frobenius operator

Let E be a espace of continuous functions $\varphi : \Lambda \to \mathbb{R}$. The Ruelle-Perron-Frobenius operator $\mathcal{L} : E \to E$ is given by

$$\mathcal{L}(\varphi)(y) = \varphi(f^{-1}(y))e^{\phi(f^{-1}(y))},$$

for some continuous potential ϕ. In our case of entropy maximal measure, we will consider always the potential $\phi \equiv 0$. A stable leaf is a subset γ of Λ with the shape $\Pi_{\Lambda}^{-1}(y)$ where $y \in N$. The corresponding foliations is denoted by $\mathcal{F}^s$.

Mass distribution

There exists a family of measures $\{\mu_\gamma\}_{\gamma \in \mathcal{F}^s}$ in Λ, such that $\forall \gamma \in \mathcal{F}^s$ and $\gamma_j \in \mathcal{F}^s$, where $f(\gamma_j) \subset \gamma$

$$\mu_\gamma(A) = \frac{1}{p} \sum_{j=1}^{p} \mu_{\gamma_j}(f^{-1}(A))$$

holds and for all continuous $\varphi : \Lambda \to \mathbb{R}$

$$\int_{f(\gamma_j)} \varphi d\mu_\gamma = \frac{1}{p} \int_{\gamma_j} \varphi \circ f d\mu_{\gamma_j}.$$

Writing $\rho_j := \frac{1}{p} \rho \circ f e^{\phi}$ then

$$\int_{\gamma} \mathcal{L}(\varphi) \rho d\mu_\gamma = \sum^{p} \int_{\gamma_j} \varphi \rho_j d\mu_{\gamma_j}.$$

If $\sup \phi - \inf \phi < \varepsilon$ and $\left| e^{\phi} \right|_{\alpha} \leq \varepsilon \inf e^{\phi}$:

Lemma

There exist $0 < \lambda < 1$ and $\kappa > 0$ such that :

1. *If $\rho \in \mathcal{D}(\gamma, \kappa)$ then $\rho_j \in \mathcal{D}(\gamma_j, \lambda\kappa)$ for all $j \in \{1, \dots, p\}$.*
2. *For all $\gamma \in \mathcal{F}^s_{loc}$, if $\rho, \hat{\rho} \in \mathcal{D}(\gamma, \lambda\kappa)$ then* $\theta(\rho, \hat{\rho}) \leq 2 \log \left(\frac{1+\lambda}{1-\lambda} \right)$.
3. *If $\rho', \rho'' \in \mathcal{D}(\gamma, \kappa)$ then there exists $\Lambda_1 = 1 - \left(\frac{1-\lambda}{1+\lambda} \right)^2$ such that $\theta_j(\rho'_j, \rho''_j) \leq \Lambda_1 \theta(\rho', \rho'')$ for all $j \in \{1, \dots, p\}$;*

θ_j and θ: projetive metrics associated to cones $\mathcal{D}(\gamma_j, \kappa)$, $\mathcal{D}(\gamma, \kappa)$

Main Cone

Let $C(b, c, \alpha)$ be a cone of functions $\varphi \in E$, such that given $\gamma \in \mathcal{F}^s$ we have:

(A) For all $\rho \in \mathcal{D}(\gamma, \kappa)$:

$$\int_\gamma \varphi \rho d\mu_\gamma > 0$$

(B) For all $\rho', \rho'' \in \mathcal{D}_1(\gamma)$:

$$\left| \int_\gamma \varphi \rho' d\mu_\gamma - \int_\gamma \varphi \rho'' d\mu_\gamma \right| < b\theta\left(\rho', \rho''\right) \inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_\gamma \varphi \rho d\mu_\gamma \right\}$$

(C) Given any leaf $\widetilde{\gamma}$ sufficiently close to γ:

$$\left| \int_\gamma \varphi d\mu_\gamma - \int_{\widetilde{\gamma}} \varphi d\mu_{\widetilde{\gamma}} \right| < cd(\gamma, \widetilde{\gamma})^\alpha \inf_\gamma \left\{ \int_\gamma \varphi d\mu_\gamma \right\}$$

Main Cone

Lemma

$C(b,c,\alpha)$ is a projective cone, that is, a convex cone such that

$$\overline{C(b,c,\alpha)} \cap -\overline{C(b,c,\alpha)} = 0.$$

Proposition

There exists $0 < \sigma < 1$ such that $\mathcal{L}(C(b,c,\alpha)) \subset C(\sigma b, \sigma c, \alpha)$ for sufficiently big b and c.

Idea of the proof: condition (B)

Condition (A):

$$\int_{\gamma} \mathcal{L}(\varphi)\rho d\mu_{\gamma} = \sum_{j=1}^{p} \int_{\gamma_j} \varphi \rho_j d\mu_{\gamma_j}$$

Condition (B):

$$\frac{\left| \int_{\gamma} \mathcal{L}(\varphi)\rho' d\mu_{\gamma} - \int_{\gamma} \mathcal{L}(\varphi)\rho'' d\mu_{\gamma} \right|}{\inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_{\gamma} \mathcal{L}(\varphi)\rho d\mu_{\gamma} \right\} \theta(\rho', \rho'')}$$

is bounded by

$$\left(1 - \left(\frac{1-\lambda}{1+\lambda}\right)^2 + 2\log\left(\frac{1+\lambda}{1-\lambda}\right)^2\right)\left(1 + \kappa diam M^{\alpha}\right)^2 b + 2M(\kappa, \alpha)$$

Idea of proof: condition (C)

Condition (C): For ϕ constant it follows that

$$\inf_{\gamma}\left\{\int_{\gamma}\mathcal{L}\varphi d\mu_{\gamma}\right\} \geq e^{\phi}\inf_{\gamma}\left\{\int_{\gamma}\varphi d\mu_{\gamma}\right\}.$$

We can assume that

$$d(\gamma_1,\widetilde{\gamma}_1) \leq \lambda_u d(\gamma,\widetilde{\gamma})$$

and in the other cases

$$d(\gamma_j,\widetilde{\gamma}_j) \leq L d(\gamma,\widetilde{\gamma})$$

for sufficiently big $0 < \lambda_u < 1$ and $L > 1$ close to 1.

Ideas in the proof: condition (C)

We have the following bound:

$$\frac{\left|\int_{\gamma}\mathcal{L}\varphi d\mu_{\gamma}-\int_{\widetilde{\gamma}}\mathcal{L}\varphi d\mu_{\widetilde{\gamma}}\right|}{d(\gamma,\widetilde{\gamma})^{\alpha}\inf_{\gamma}\left\{\int_{\gamma}\mathcal{L}\varphi d\mu_{\gamma}\right\}} \leq \underbrace{\frac{\lambda_u^{\alpha}+(p-1)(1+(L-1)^{\alpha})L^{\alpha}}{p}}_{<1,\text{ since } L\simeq 1} c.$$

Finite diameter of the main cone

Proposition

For all $b > 0$, $c > 0$ *and* $\alpha \in (0,1]$ *it follows*

$$\Delta := \sup\{\Theta(\mathcal{L}\varphi, \mathcal{L}\psi); \varphi, \psi \in C(b,c,\alpha)\} < \infty.$$

Finite diameter of the main cone

Analizing the condition (A),(B) and (C) we obtain

$$\alpha(\varphi,\psi) = \inf \left\{ \frac{\int_\gamma \psi\rho d\mu_\gamma}{\int_\gamma \varphi\rho d\mu_\gamma}, \frac{\int_\gamma \psi\hat{\rho} d\mu_\gamma}{\int_\gamma \varphi\hat{\rho} d\mu_\gamma}\xi(.), \frac{\int_{\hat{\gamma}} \psi d\mu_{\hat{\gamma}}}{\int_{\hat{\gamma}} \varphi d\mu_{\hat{\gamma}}}\eta(.) \right\}$$

and

$$\beta(\varphi,\psi) = \sup \left\{ \frac{\int_\gamma \varphi\rho d\mu_\gamma}{\int_\gamma \psi\rho d\mu_\gamma}, \frac{\int_\gamma \varphi\hat{\rho} d\mu_\gamma}{\int_\gamma \psi\hat{\rho} d\mu_\gamma}\xi(.), \frac{\int_{\hat{\gamma}} \varphi d\mu_{\hat{\gamma}}}{\int_{\hat{\gamma}} \psi d\mu_{\hat{\gamma}}}\eta(.) \right\}.$$

Finite diameter of the main cone

Where $\xi(\gamma, \rho', \rho'', \hat{\rho}, \varphi, \psi)$ is equal to

$$\frac{\left(\int_\gamma \psi\rho'' d\mu_\gamma - \int_\gamma \psi\rho' d\mu_\gamma\right) / \int_\gamma \psi\hat{\rho} d\mu_\gamma + b\theta(\rho', \rho'')}{\left(\int_\gamma \varphi\rho'' d\mu_\gamma - \int_\gamma \varphi\rho' d\mu_\gamma\right) / \int_\gamma \varphi\hat{\rho} d\mu_\gamma + b\theta(\rho', \rho'')}$$

and

$$\eta(\gamma, \widetilde{\gamma}, \hat{\gamma}, \hat{\rho}, \varphi, \psi) = \frac{\left(\int_\gamma \psi d\mu_\gamma - \int_{\widetilde{\gamma}} \psi d\mu_{\widetilde{\gamma}}\right) / \int_{\hat{\gamma}} \psi d\mu_{\hat{\gamma}} + cd(\gamma, \widetilde{\gamma})}{\left(\int_\gamma \varphi d\mu_\gamma - \int_{\widetilde{\gamma}} \varphi d\mu_{\widetilde{\gamma}}\right) / \int_{\hat{\gamma}} \varphi d\mu_{\hat{\gamma}} + cd(\gamma, \widetilde{\gamma})}.$$

Finite diameter of the main cone

Given $\varphi, \psi \in C(\sigma b, \sigma c, \alpha)$ we note that

$$\frac{1-\sigma}{1+\sigma} < \xi(\gamma, \rho', \rho'', \hat{\rho}, \psi, \varphi) < \frac{1+\sigma}{1-\sigma}$$

and the same for $\eta(\gamma, \widetilde{\gamma}, \hat{\gamma}, \hat{\rho}, \psi, \varphi)$. Denoting by Θ_+ projective metric associated to the cone defined by the condition (A) , then

$$\Theta_+(\varphi, \psi) = \sup_{\gamma, \rho \in \mathcal{D}(\gamma), \hat{\gamma}, \hat{\rho} \in \mathcal{D}(\hat{\gamma})} \left\{ \frac{\int_\gamma \varphi \rho d\mu_\gamma \int_{\hat{\gamma}} \psi \hat{\rho} d\mu_{\hat{\gamma}}}{\int_{\hat{\gamma}} \varphi \hat{\rho} d\mu_{\hat{\gamma}} \int_\gamma \psi \rho d\mu_\gamma} \right\}$$

It follows that

$$\Theta(\varphi, \psi) < \Theta_+(\varphi, \psi) + log \left(\frac{1+\sigma}{1-\sigma} \right)^2.$$

Finite diameter of the main cone

We just need to see that $\{\Theta_+(\mathcal{L}\varphi, 1); \varphi \in C(b, c, \alpha)\}$ is uniformly bounded in φ:

$$\frac{\displaystyle\int_{\hat{\gamma}} \mathcal{L}\varphi\hat{\rho}d\mu_{\hat{\gamma}}}{\displaystyle\int_{\gamma} \mathcal{L}\varphi\rho d\mu_{\gamma}} = \frac{\displaystyle\sum_{j=1}^{p}\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\displaystyle\sum_{j=1}^{p}\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}}$$

where $\varphi \in C(b, c, \alpha)$, $\rho \in \mathcal{D}_1(\gamma)$ and $\hat{\rho} \in \mathcal{D}_1(\hat{\gamma})$.

Finite diameter of the main cone

All we need is to bound

$$\frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}} = \frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\int_{\hat{\gamma}_j} \varphi d\mu_{\hat{\gamma}_j}} \frac{\int_{\hat{\gamma}_j} \varphi d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi d\mu_{\gamma_j}} \frac{\int_{\gamma_j} \varphi d\mu_{\gamma_j}}{\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}}.$$

Noting that

$$\frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi\rho_j d\mu_{\gamma_j}} < \left(1 + b\log\left(\frac{1+\lambda}{1-\lambda}\right)\right)^2 (1+\max\{\kappa, c, \varepsilon\} diam(M)^\alpha)^4,$$

we end the proof of the finite diameter of the main cone.

Exponential decay of correlations

We first prove that $\mu = \mu_\gamma \times \nu$, where

$$\mu_\gamma \times \nu(\varphi) := \int \int_\gamma \varphi d\mu_\gamma d\nu.$$

Consider the dual operator $\tilde{\mathcal{L}}^*$, such that

$$\int \tilde{\mathcal{L}}\varphi d\mu = \int \varphi d\tilde{\mathcal{L}}^* \mu.$$

Exponential decay of correlations

For μ

$$\int (\varphi \circ f^n)\, \psi d\mu = \int \varphi \tilde{\mathcal{L}}^n(\psi) d\mu \tag{2}$$

holds.

Now we are able to prove that for Hölder observables ϕ, ψ, there exists $0 < \tau < 1$ and $K(\varphi, \psi) > 0$ such that

$$\left| \int \varphi \tilde{\mathcal{L}}^n\, (\psi)\, d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq K(\varphi, \psi) \cdot \tau^n, \quad \text{for all } n \geq 1.$$

Exponential decay of correlations

For all $\gamma \in \mathcal{F}^s_{loc}$

$$\frac{\displaystyle\int_\gamma \varphi \tilde{\mathcal{L}}^n(\psi) d\mu_\gamma}{\displaystyle\int_\gamma \varphi d\mu_\gamma} \leq \frac{\beta_+\left(\tilde{\mathcal{L}}^n(\psi), 1\right)}{\alpha_+\left(\tilde{\mathcal{L}}^n(\psi), 1\right)} \leq e^{\Theta_+\left(\tilde{\mathcal{L}}^n(\psi),1\right)} \leq e^{\Theta\left(\tilde{\mathcal{L}}^n(\psi),1\right)}.$$

Thus,

$$\frac{\displaystyle\int \varphi \tilde{\mathcal{L}}^n(\psi) d\mu}{\displaystyle\int \varphi d\mu} = \frac{\displaystyle\int\int_\gamma \varphi \tilde{\mathcal{L}}^n(\psi) d\mu_\gamma d\nu}{\displaystyle\int\int_\gamma \varphi d\mu_\gamma d\nu} \leq e^{\Theta\left(\tilde{\mathcal{L}}^n(\psi),1\right)} \leq e^{\Delta \tau^{n-1}}.$$

Exponential decay of correlations

This permits us to prove

$$\left|\int \varphi d\mu\right| \left|\frac{\int \varphi \tilde{\mathcal{L}}^n(\psi) d\mu}{\int \varphi d\mu} - 1\right| \leq \left|\int \varphi d\mu\right| \left(e^{\Delta \tau^{n-1}} - 1\right) \leq K(\varphi)\tau^n.$$

For $\int \psi d\mu \neq 1$: $K(\varphi, \psi) = \left|\int \psi d\mu\right| K(\varphi)$.

Theorem

(Central Limit Theorem)
Given φ a Hölder continuous function

$$\sigma_\varphi^2 := \int \phi^2 d\mu + 2\sum_{j=1}^{\infty} \int \phi \cdot (\phi \circ f^j)\, d\mu, \quad \text{onde} \quad \phi = \varphi - \int \varphi\, d\mu.$$

Then $\sigma_\varphi < \infty$ and $\sigma_\varphi = 0$ if, and only if, $\varphi = u \circ f - u$ para algum $u \in L^1(\mu)$. Moreover, if $\sigma_\varphi > 0$ then for all interval $A \subset \mathbb{R}$

$$\lim_{n\to\infty} \mu\left(A_\varphi^n\right) = \frac{1}{\sigma_\varphi\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\sigma_\varphi^2}} dt.$$

holds, where $A_\varphi^n = \left\{ x \in M : \frac{1}{\sqrt{n}} \sum^{n-1} \left(\varphi(f^j(x)) - \int \varphi d\mu\right) \in A \right\}$

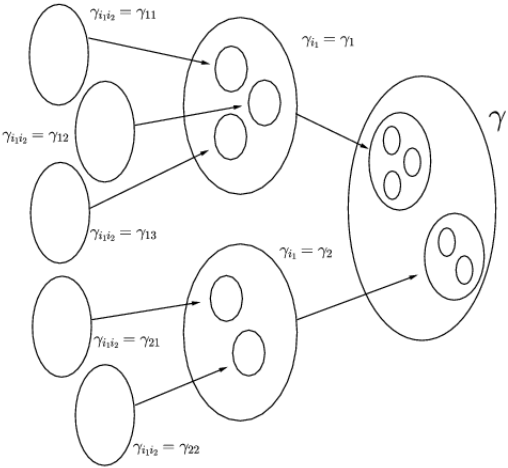


Figura: Mass distribution for derived from Anosov

Invariant Cones for the Second Setting

Given $n \in \mathbb{N}$, $n \geq 1$, we can write

$$\gamma = \bigcup_{i_1=1}^{p_{i_0}} \bigcup_{i_2=1}^{p_{i_1}} \cdots \bigcup_{i_n=1}^{p_{i_{n-1}}} f^n(\gamma_{i_1 \cdots i_n})$$

where $f(\gamma_{i_1 \cdots i_{n+1}}) \subset \gamma_{i_1 \cdots i_n}$, $\gamma_{i_1 \cdots i_n} = \bigcup_{i_{n+1}=1}^{p_{i_n}} f(\gamma_{i_1 \cdots i_{n+1}})$ e $p_{i_k}, k \in \mathbb{N}$.

Note also that $p_{i_0} = p_\gamma$.

Invariant Cones for the Second Setting

For each strong stable leaf γ we define a probability measure by:

$$\mu_\gamma\left(f^n\left(\gamma_{i_1\cdots i_n}\right)\right) := \frac{1}{p_{i_0}p_{i_1}\cdots p_{i_{n-1}}}.$$

We also write

$$\int_\gamma \mathcal{L}(\varphi)\rho d\mu_\gamma = \sum_{j=1}^{p_\gamma}\int_{\gamma_j}\varphi\rho_j d\mu_{\gamma_j}$$

where $\rho_j := \frac{1}{p_\gamma}\rho\circ f e^{\phi}$.

Invariant Cones for the Second Setting

Let $C[b,c,\alpha]$ the cone of functions $\varphi \in E$ s.t. for $\gamma \in \mathcal{F}^s$ we have

- (A) For all $\rho \in \mathcal{D}(\gamma,\kappa)$:

$$\int_\gamma \varphi\rho d\mu_\gamma > 0$$

- (B) For all $\rho', \rho'' \in \mathcal{D}_1(\gamma)$:

$$\left|\int_\gamma \varphi\rho' d\mu_\gamma - \int_\gamma \varphi\rho'' d\mu_\gamma\right| < b\theta\left(\rho',\rho''\right) \inf_{\rho\in\mathcal{D}_1(\gamma)}\left\{\int_\gamma \varphi\rho d\mu_\gamma\right\}$$

- (C) For all stable leaves γ and $\tilde{\gamma}$ in the same Markovian rectangle R_i:

$$\left|\int_\gamma \varphi d\mu_\gamma - \int_{\tilde{\gamma}} \varphi d\mu_{\tilde{\gamma}}\right| < cd(\gamma,\tilde{\gamma})^\alpha \inf_\gamma\left\{\int_\gamma \varphi d\mu_\gamma\right\}$$

Invariance and Finite Diameter Results

Proposition

There exists $0 < \sigma < 1$ *such that* $\mathcal{L}(C[b, c, \alpha]) \subset C[\sigma b, \sigma c, \alpha]$ *suficiently big* b *and* c.

Proposition

For all $b > 0$, $c > 0$ *and* $\alpha \in (0, 1]$ *the diameter of* $\mathcal{L}(C[b, c, \alpha])$ *is finite, that is, there exists*
$\Delta := \sup \{\Theta(\mathcal{L}\varphi, \mathcal{L}\psi); \varphi, \psi \in C[b, c, \alpha]\} < \infty$.

Finite diameter in the Second Setting

The proof of the finite diameter of the image of the cone in the Second Setting is more subtle, and uses the mild mixing assumption that we have done on the Markov Partition

$$\mathcal{R} = \{R_1, \cdots, R_p\}, p \geq 2.$$

Finite diameter in the Second Setting

As before, we obtain that

$$\Theta(\mathcal{L}(\varphi), \mathcal{L}(\psi)) < \Theta_+(\mathcal{L}(\varphi), \mathcal{L}(\psi)) + \mathit{log} \left(\frac{1+\sigma}{1-\sigma}\right)^2.$$

Finite diameter in the Second Setting

As we have seen, in order to bound $\Theta_+(\mathcal{L}(\varphi), \mathcal{L}(\psi))$, one just need to bound

$$\frac{\int_{\hat{\gamma}} \mathcal{L}\varphi\hat{\rho}d\mu_{\hat{\gamma}}}{\int_{\gamma} \mathcal{L}\varphi\rho d\mu_{\gamma}} = \frac{\sum_{k=1}^{p}\sum_{l=1}^{r_k(\hat{\gamma})}\int_{\hat{\gamma}_{k_l}} \varphi(\hat{\rho})_{k_l} d\mu_{\hat{\gamma}_{k_l}}}{\sum_{k=1}^{p}\sum_{l=1}^{r_k(\gamma)}\int_{\gamma_{k_l}} \varphi\rho_{k_l} d\mu_{\gamma_{k_l}}},$$

where the inner sums are over leaves in the same rectangle of the Markov Partition.

Finite diameter in the Second Setting

Thus, for each such inner sumation there exists $\tilde{C}$ such that

$$\frac{\displaystyle\sum_{l=1}^{r_k(\hat{\gamma})} \int_{\hat{\gamma}_{k_l}} \varphi(\hat{\rho})_{k_l} \, d\mu_{\hat{\gamma}_{k_l}}}{\displaystyle\sum_{l=1}^{r_k(\gamma)} \int_{\gamma_{k_l}} \varphi \rho_{k_l} \, d\mu_{\gamma_{k_l}}} \leq \tilde{C},$$

and this permits us to bound $\Theta_+(\mathcal{L}(\varphi), \mathcal{L}(\psi))$.

Construction of the measure in the Second Setting

Define the equivalence relation: $x \sim y$ iff, x and y are in the same leaf $\gamma \in \mathcal{F}^s_{loc}$. Let $g : \mathcal{F}^s_{loc} \to \mathcal{F}^s_{loc}$ defined by $g(\widetilde{x}) = \widetilde{f(x)}$ and $\hat{\eta}$ a $g-$invariant probability measure. Consider

$$\eta(\varphi) := \int \left(\int_\gamma \varphi d\mu_\gamma \right) d\hat{\eta}(\gamma).$$

Proposition

Let $(\varphi_n)_n$ be a sequence in $C[b, c, \alpha]$, Θ_+-Cauchy normalized by $\int \varphi_n d\eta = 1$ and a continuous function $\psi : \Lambda \to \mathbb{R}$. Then, the sequence $\left(\int \varphi_n \psi d\eta \right)_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$.

Given $\varphi \in C[b, c, \alpha]$ define

$$\mu_\varphi(\psi) := \lim_{n\to\infty} \int \varphi_n \psi d\eta.$$

where $\varphi_n = \dfrac{\mathcal{L}^n(\varphi)}{\int \mathcal{L}^n(\varphi) d\eta}$. Such functional does not depend on the choice of elements in $C[b, c, \alpha]$. Denoting $\mu_1(\psi)$ by μ, we have

$$\mu(\psi) = \int \left(\int_\gamma \psi d\mu_\gamma \right) d\hat{\eta}(\gamma).$$

Statistical Properties in the Second Setting

The Exponential Decay of Correlations and the Central Limit Theorem follows in the same manner as in the first setting.

Some further questions and work in progress:

Question 0: Other (low variation) potentials? Same results for other equilibrium states?

Question 1: Differentiability of the measure (linear response Formula)?

Question 2: Large Deviation Statements?

Question 3: Differentiability of Decay of Correlation and Large Deviation Rates, Exponents of Liapunov, etc...

(Joint work with C. Liverani, in final phase)

Remark

Such questions were positively answered in the non-uniformly expanding case by Bomfim-Castro-Varandas (preprint 2014).

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Thank you for coming!