Ergodic Properties of the Kusuoka measure

Mark Pollicott

Warwick University

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  However, a somewhat different approach has to be introduced in the proof.

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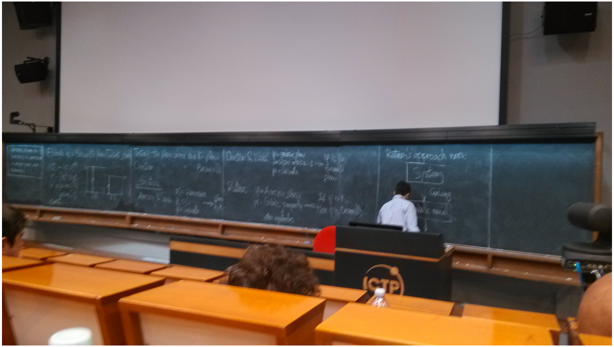

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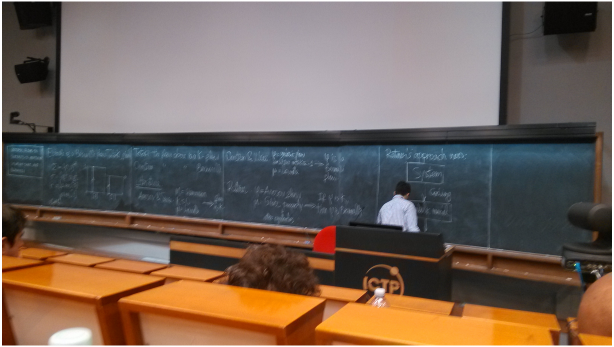

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In this talk we will discuss Gibbs measures for a very simple uniformly hyperbolic system and **non-Hölder continuous** (discontinuous) potentials.

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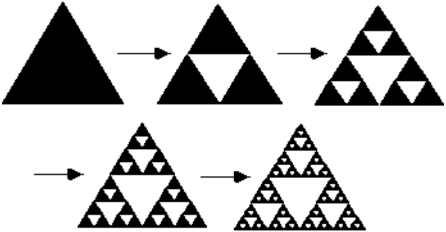

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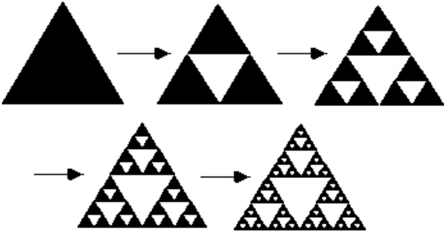

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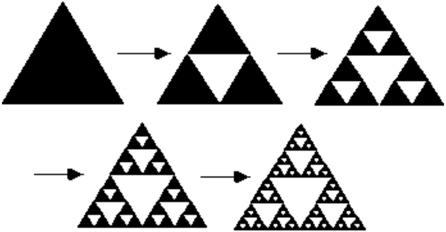

- Replace a triangle in the plane by the three triangles of half the size in the corners.
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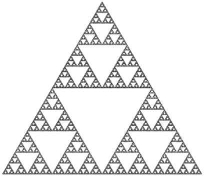
- We continue this process iteratively to get the "fractal" X.



Waclaw Sierpiński

Waclaw Sierpiński (1882-1969) was a distinguished polish number theorist and set theorist. In 1951 the Warsaw Scientific Society issued a medal in his honour.

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$$d\left((x_n)_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\right) = \sum_{n=0}^{\infty} \frac{e\left(x_n, y_n\right)}{2^n} \text{ where } e(i,j) = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

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where $e_1 = (0,0)$, $e_2 = (1,0)$ and $e_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, say.

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- We can introduce some dynamics by $\sigma : \Sigma \to \Sigma$ the usual shift map given by

$$\sigma(x_n)_{n=0}^{\infty} = (x_{n+1})_{n=0}^{\infty}.$$

  (This essentially corresponds to a map on X which doubles distances).

Measures

The coding gives a convenient viewpoint for studying probability measures on X, by considering measures on Σ. Recall that μ is σ-invariant if

$$\mu([x_0, \cdots, x_{n-1}]) = \sum_{i=1}^{3} \mu([i, x_0, \cdots, x_{n-1}])$$

for all cylinders $[x_0, \cdots, x_{n-1}] := \{y = (y_n)_{n=0}^{\infty} \in \Sigma : x_j = y_j \text{ for } 0 \leq j \leq n-1\}$ where $x_0, \cdots, x_{n-1} \in \{1, 2, 3\}$.

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Example (Most obvious example)

The $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on Σ satisfies $\mu([x_0, \cdots, x_{n-1}]) = \frac{1}{3^n}$ and corresponds to the "natural" measure on X.

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Similarly, one could take Gibbs measures (for Hölder potentials $\psi : \Sigma \to \mathbb{R}$).

Definition

A σ-invariant measure μ is a Gibbs measure (for the potential $\log \psi$) if

$$\psi(x) = \lim_{n \to +\infty} \frac{\mu[x_0, \cdots, x_n]}{\mu[x_1, \cdots, x_n]}$$

satisfies that $\psi : \Sigma \to \mathbb{R}$ is Hölder continuous.

Classical examples

Example (Obvious example revisited)

If μ is the $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on Σ then $\mu[x_0, \cdots, x_{n-1}] = \frac{1}{3^n}$ and $\log \psi(x) = -\log 3$ is a constant function.

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If μ is the (p_1, p_2, p_3)-Bernoulli measure on Σ (with $p_1 + p_2 + p_3 = 1$) then $\mu([x_0, \cdots, x_{n-1}]) = p_{x_0} p_{x_1} \cdots p_{x_{n-1}}$ and for $x = (x_n)_{n=0}^{\infty}$:

$$\log \psi(x) = \begin{cases} \log p_1 & \text{if } x_0 = 1 \\ \log p_2 & \text{if } x_0 = 2 \\ \log p_3 & \text{if } x_0 = 3 \end{cases}$$

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More generally, in the standard Gibbs theory approach one likes the potential $\log \psi$ to be Hölder continuous.

However, the Kusuoka measure is defined in a different sort of way and has a different kind of potential ...

Kusuoka measure

The Kusuoka measure was originally defined on the Sierpiński triangle X, but to describe the corresponding measure μ on Σ we want to specify the measure of cylinder sets

$$[i_0, \cdots, i_{n-1}] = \{x = (x_k)_{k=0}^{\infty} : x_j = i_j \text{ for } 0 \leq j \leq n-1\}, \text{ for } i_0, \cdots, i_{n-1} \in \{1, 2, 3\}.$$

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For concreteness, let us begin with the classical case.

Definition (Classical Kusuoka measure)

Let $A_1 = \begin{pmatrix} \frac{3}{\sqrt{15}} & 0 \\ 0 & \frac{1}{\sqrt{15}} \end{pmatrix}$, $A_2 = \frac{\sqrt{5}}{2} \begin{pmatrix} \frac{\sqrt{3}}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{\sqrt{3}} \end{pmatrix}$ and $A_3 = \frac{\sqrt{5}}{2} \begin{pmatrix} \frac{\sqrt{3}}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{1}{\sqrt{3}} \end{pmatrix}$.
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$$\mu([i_0, \cdots, i_{n-1}]) = \mathrm{trace}\left((A_{i_0} \cdots A_{i_{n-1}})^T \mathcal{E} (A_{i_0} \cdots A_{i_{n-1}}) \right) \text{ for } i_0, \cdots, i_{n-1} \in \{1, 2, 3\}$$

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The corresponding measure on X is important in defining the "Laplacian" on the fractal.

The potential for the Kusuoka measure

We can attempt to define the "potential"

$$\psi(x) = \lim_{n \to +\infty} \frac{\mu[x_0, \cdots, x_n]}{\mu[x_1, \cdots, x_n]}.$$

If $\psi : \mathbf{\Sigma} \to \mathbb{R}$ were Hölder continuous then we could apply general ideas from "theormodynamical formalism".

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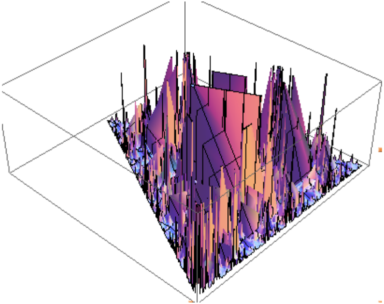
Theorem (Bell-Ho-Strichartz, 2014)

There exist a dense set of discontinuities for $\psi(x)$.



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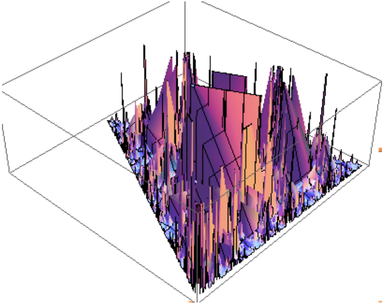

Despite this, it is possible to establish familiar ergodic properties

Main result

We can prove stronger ergodic results, such as exponential mixing:

Theorem (Johansson-Öberg-P.)

The measure μ mixed exponentially fast, i.e., there exists $0 < \alpha < 1$ such that for Lipschitz $f_1, f_2 : \Sigma \to \mathbb{R}$ we can find $C > 0$ with

$$\left| \int f_1 \circ \sigma^n . f_2 d\mu - \int f_1 d\mu . \int f_2 d\mu \right| \leq C\alpha^n$$

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Anders Öberg and Anders Johansson (explaining something very patiently to a coauthor)

Applications: Summary

The ergodicity of the Kusuoka measure gives the

Theorem (Birkhoff ergodic theorem)

For any $f \in L^1(\Sigma, \mu)$ we have that for a.e. (μ) $x \in \Sigma$, $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\sigma^k x) = \int f d\mu$.

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1. Central Limit Theorems
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3. Pointwise error terms in the Birkhoff ergodic theorem.

1. Central Limit Theorems

As we observed, ergodicity of the measure μ implies that:

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For any $L^1(\Sigma, \mu)$ function $f : \Sigma \to \mathbb{R}$ we have that

$$\frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n x) \to \int f d\mu \text{ as } N \to +\infty,$$

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The central limit theorem gives stronger results where $1/N$ is replaced by $1/\sqrt{N}$.

Theorem (Central Limit Theorm)

Assume $f : \Sigma \to \mathbb{R}$ is a Lipschitz function not cohomologous to a constant (i.e., $f - \int f d\mu = u \circ \sigma - u$ where $u \in B$). Then there exists $\sigma^2 > 0$ such that we have that for any $\alpha < \beta$ we have

$$\mu\left(\left\{x \in \Sigma : \alpha \leq \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(\sigma^n x) - \int f d\mu \leq \beta\right\}\right) \to \frac{1}{\sqrt{2\pi}\sigma} \int_\alpha^\beta e^{-\sigma^2 u^2/2} du$$

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2. Large Deviation results

Recall that:

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Another form of generalisation of the Birkhoff theorem is the following.

Theorem (Large Deviation Theorem)

Let $f : \Sigma \to \mathbb{R}$ be Lipschitz. For each $\epsilon > 0$ there exists $C > 0$, $0 < \rho < 1$ such that

$$\mu\left(\left\{x \in \Sigma : \left|\frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n x) - \int f d\mu\right| > \epsilon\right\}\right) \le C\rho^n$$

as $N \to +\infty$.

There is also be a corresponding version for measures $\frac{1}{N} \sum_{n=0}^{N-1} \delta_{\sigma^n x}$.

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In particular, for any $\epsilon > 0$ can write

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The strategy of the proof

Let us define a function $g : X \to \mathbb{R}$ by

$$\psi(x) = \lim_{n \to +\infty} \frac{\mu[x_0, \cdots, x_{n-1}]}{\mu[x_1, \cdots, x_{n-1}]} \text{ for } a.e.(\mu) x = (x_n)_{n=0}^{\infty} \in \Sigma.$$

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- to prove there is a spectral gap for L.

The space of functions B

Let $\mathcal{A}_n$ ($n \geq 0$) be the finite sigma algebra consisting of all cylinders $[i_0, \cdots, i_{n-1}]$ of length n (N.B. those traditionally used in the definition of entropy).

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Given $0 < \theta < 1$, let

$$B = B_\theta := \left\{ f : \|f\|_\theta^2 := \sum_{n=1}^{\infty} \frac{\|\mathbb{E}(f|\mathcal{A}_n) - \mathbb{E}(f|\mathcal{A}_{n-1})\|_2^2}{\theta^n} < +\infty \right\}$$

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Providing $\theta > \frac{1}{2}$ we have that B contains the Lipschitz functions.

The spectrum and an indirect approach

There is an operator theorem has a reassuringly familiar statement

Theorem

For $0 < \theta < 1$ *sufficiently large,* $L : B \to B$ *defined by* $Lf(x) = \sum_{\sigma y = x} g(y) f(y)$ *is well defined. Moreover,*

- $L(1) = 1$ *(i.e., preserves the constant functions* $\mathbb{C}$*)*
- *The spectral radius of* $L : B/\mathbb{C} \to B/\mathbb{C}$ *is strictly smaller than* 1

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Unfortunately, we couldn't get that to work - so there is a more indirect approach working with Banach spaces of matrix valued functions, and operators on these (which then project down to operators on functions of the above form).

Generalizations

The Kusuoka measure is a special case of a more general class of invariant measures on Σ.

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Given any full shift $\Sigma = \{1, \cdots, k\}^{\mathbb{Z}_+}$ assume that we have:

- $d \times d$ matrices $A_1, \cdots, A_k$; and
- a positive definite $d \times d$ smatrix $\mathcal{E}$,

(for some $d \geq 1$) which satisfy

$$\sum_{i=1}^{k} A_i A_i^T = I \text{ and } \sum_{i=1}^{k} A_i^T \mathcal{E} A_i = I.$$

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and a strong irreducibility condition. We can define a *(generalized) Kusuoka measure* on Σ using

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Theorem (Johansson-Öberg-P.)

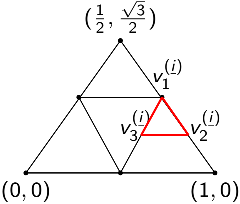
The measure μ mixed exponentially fast, i.e., there exists $0 < \alpha < 1$ such that for Lipschitz $f_1, f_2 : \Sigma \to \mathbb{R}$ we can find $C > 0$ with

$$\left| \int f_1 \circ \sigma^n . f_2 d\mu - \int f_1 d\mu . \int f_2 d\mu \right| \leq C\alpha^n$$

Aside: The origins of the Kusuoka measusre

Question

Why was the Kusuoka measure introduced?



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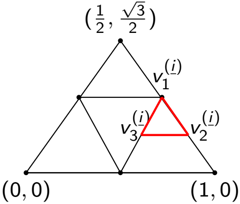

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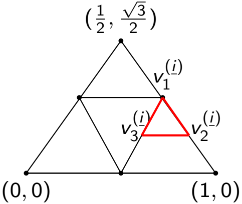

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Definition

Given $u \in C^0(X, \mathbb{R})$ we define the *energy* in terms of the values on smaller triangles (with vertices $v_1^{(\underline{i})}$, $v_2^{(\underline{i})}$, $v_3^{(\underline{i})}$ corresponding to cylinders $[\underline{i}] = [i_0, \cdots, i_{n-1}]$) in graphs approximating the fractal X.

$$\mathcal{E}(u) := \lim_{n \to +\infty} \sum_{\underline{i} \in \{1,2,3\}^n} \sum_{1 \le r < s \le 3} \left(u\left(v_r^{(\underline{i})}\right) - u\left(v_s^{(\underline{i})}\right)\right)^2 \in [0, +\infty]$$

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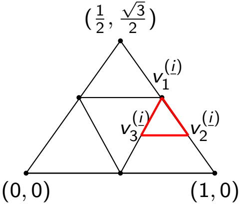

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and $\mathcal{E}(u, v) = \frac{1}{4} \left(\mathcal{E}(u+v) - \mathcal{E}(u-v) \right)$ for $u, v \in C^0(X, \mathbb{R})$

Harmonic functions and Harmonic measures

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there is a unique function $u \in C(X, \mathbb{R})$ achieving these three values and minimizing $\mathcal{E}(u)$. This is called a *harmonic function*.

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Definition

We can associated to a harmonic function $u \in C(X, \mathbb{R})/\mathbb{R}$ a *harmonic measure* ν_u on X by

$$\nu_u(\pi([i_0, \cdots, i_{n-1}])) = \left(\frac{5}{3}\right)^n \mathcal{E}(u \circ \pi(i_0, \cdots, i_{n-1}, x_0, x_1, \cdots))$$

where $i_0, \cdots, i_{n-1} \in \{1, 2, 3\}^n$.

The Kusuoka measure and the laplacian

Finally, fix a basis u_1, u_2 for harmonic functions satisfying $\mathcal{E}(u_1, u_2) = 0$ and $\mathcal{E}(u_1) = \mathcal{E}(u_2) = 1$.

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The measure $\mu = \nu_{u_1} + \nu_{u_2}$.

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One defines a *Laplacian* Δ on suitable functions $f_1 \in C(X, \mathbb{R})$ by

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for suitable $f_2 \in C(X, \mathbb{R})$.

In particular, the Kusuoka measure gives the Laplacian desirable properties (that wouldn't happen with the Bernoulli measure, say).

The Kusuoka measure and the laplacian

Finally, fix a basis u_1, u_2 for harmonic functions satisfying $\mathcal{E}(u_1, u_2) = 0$ and $\mathcal{E}(u_1) = \mathcal{E}(u_2) = 1$.

Definition (Alternative equivalent definition of Kusuoka measure)

The measure $\mu = \nu_{u_1} + \nu_{u_2}$.

This is then used to define a laplacian.

Definition

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In particular, the Kusuoka measure gives the Laplacian desirable properties (that wouldn't happen with the Bernoulli measure, say). For example,

Lemma

If $\Delta f \in L^2(\mu)$ then $\Delta(f^2) \in L^2(\mu)$.

The end

Thank you for your time and attention