Complexity and Sturmian colorings of regular trees

(joint work w/ Seonhee Lim)

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Complexity of infinite words

Consider sequences in $\mathcal{A}^{\mathbb{N}} = \{\mathtt{a}, \mathtt{b}\}^{\mathbb{N}}$.

abaabaabaabaabaabaabaabaabaabaabaabaabaabaabaab . . .

abaababaabaababaababaabaababaababaabaababaabaab . . .

abaabbababaaaabbbabbaabbbababbbbbabbabaaabababb . . .

Subword complexity (Factor complexity)

The subword complexity for an infinite word $\mathbf{u} \in \mathcal{A}^{\mathbb{N}} = \{\mathtt{a}, \mathtt{b}\}^{\mathbb{N}}$

$p_{\mathbf{u}}(n)$ = the number of different subwords of length n in $\mathbf{u}$.

abaabaabaabaabaabaabaabaabaabaabaabaabaabaabaab ...

abaababaabaababaababaabaababaabaababaababaabaab ...

abaabbababaaaabbbabbaabbbababbbbbabbabaaabababb ...

Subword complexity

$p_{\mathbf{u}}(n)$ = the number of different subwords of length n in $\mathbf{u}$.

abaabaabaabaabaabaabaabaabaabaabaabaabaabaab...

$p(1) = 2$ a, b

abaababaabaababaababaabaababaababaabababaabaab...

$p(1) = 2$ a, b

abaabbababaaaabbbabbaabbbababbbbabbabaaababab b...

$p(1) = 2$ a, b

Subword complexity

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abaabaabaabaabaabaabaabaabaabaabaabaabaabaabaab ...

$p(1) = 2, \quad p(2) = 3$ ab, ba, aa

abaababaabaababaababaabaabaababaababaabaababaabaab ...

$p(1) = 2, \quad p(2) = 3$ ab, ba, aa

abaabbababaaaabbbabbaabbababbbbabbabaaabababb ...

$p(1) = 2, \quad p(2) = 4$ ab, ba, aa, bb

Subword complexity

$p_{\mathbf{u}}(n) =$ the number of different subwords of length n in $\mathbf{u}$.

abaabaabaabaabaabaabaabaabaabaabaabaabaabaabaab...

$p(1) = 2, \quad p(2) = 3, \quad p(3) = 3$ aba, baa, aab

abaababaabaababaababaababaabaababaabaababaabaab...

$p(1) = 2, \quad p(2) = 3, \quad p(3) = 4$ aba, baa, aab, bab

abaabbababaaaabbbabbaabbbababbbbbabbabaaabababb...

$p(1) = 2, p(2) = 4, p(3) = 8$ aba, bab, aaa, bba, aab, bbb, abb, baa

Subword complexity

$p_{\mathbf{u}}(n)$ = the number of different subwords of length n in $\mathbf{u}$.

abaababaabaabaabaababaabaabaabaabaabaabaabaabaabaabaab...

$p(1) = 2, \quad p(2) = 3, \quad p(3) = 3, \quad p(4) = 3$

abaabababaabaababaababaabaabababaababaabaababaabaab...

$p(1) = 2, \quad p(2) = 3, \quad p(3) = 4, \quad p(4) = 5$

abaabbabababaaaabbbabbaabbbababbbbbabbabaaababababb...

$p(1) = 2, \quad p(2) = 4, \quad p(3) = 8, \quad p(4) = 16$

Subword complexity

$p_{\mathbf{u}}(n)$ = the number of different subwords of length n in $\mathbf{u}$.

abaabaabaabaabaabaabaabaabaabaabaabaabaabaab...

$p(1) = 2, \quad p(2) = 3, \quad p(3) = 3, \quad p(4) = 3, \quad p(n) = 3$

abaababaabaababaababaabaababaabaababaabaab...

$p(1) = 2, \quad p(2) = 3, \quad p(3) = 4, \quad p(4) = 5, \quad p(n) = n + 1$

abaabbababaaaabbbabbaabbbababbbbabbabaaababab...

$p(1) = 2, \quad p(2) = 4, \quad p(3) = 8, \quad p(4) = 16, \quad p(n) = 2^n$

Subword complexity and Sturmian word

Theorem (Hedlund-Morse (1940))

The followings are equivalent:

1. *The infinite word* $\mathbf{u}$ *is eventually periodic.*
2. *The subword complexity* $p_{\mathbf{u}}$ *satisfies* $p_{\mathbf{u}}(n+1) = p_{\mathbf{u}}(n)$ *for some* $n > 0$.
3. *The subword complexity* $p_{\mathbf{u}}$ *is bounded.*

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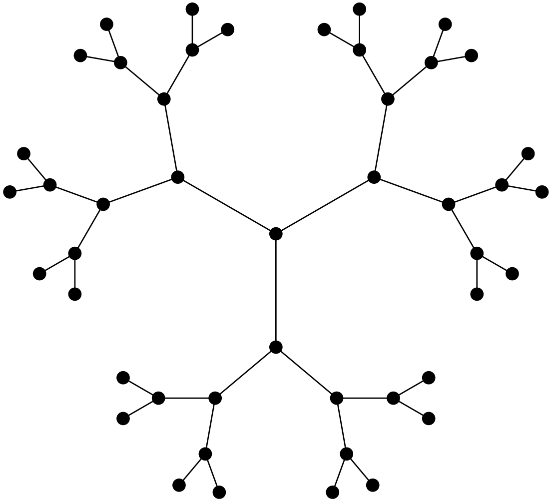
Definition

An infinite word $\mathbf{u}$ is called Sturmian if $P_{\mathbf{u}}(n) = n + 1$.

Sturmian words have the lowest complexity among non-eventually periodic words.

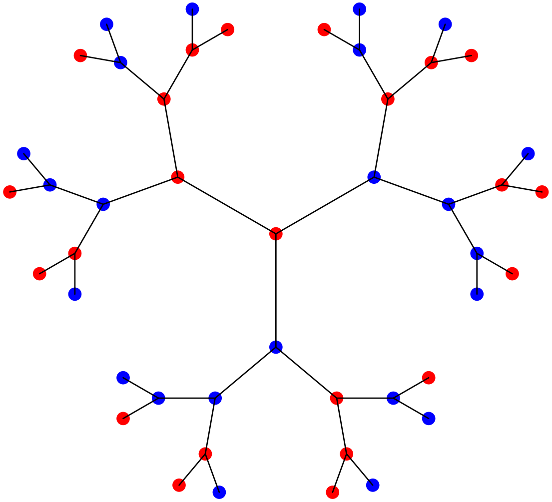
A regular tree



T is k-regular tree,
i.e., each vertex
has k edges.

All edge lengths $= 1$.

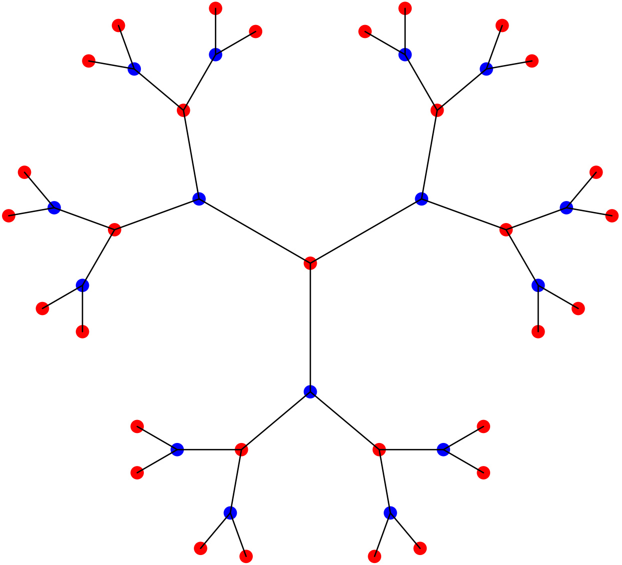
A vertex coloring of 3-regular tree



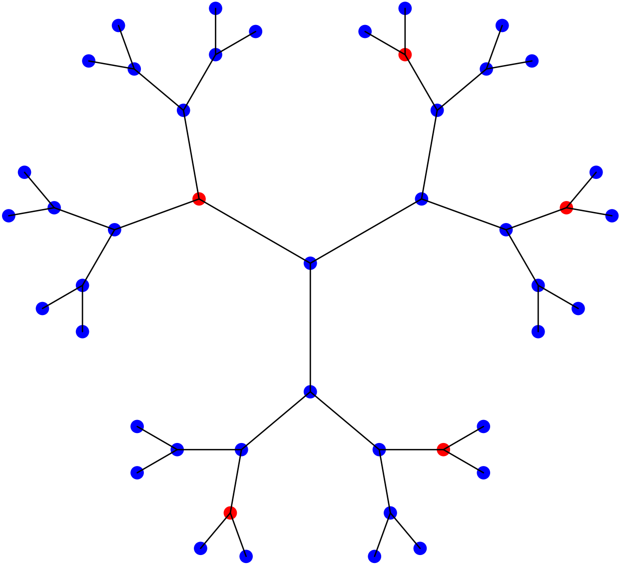
A coloring of T is a map $\phi : VT \to \mathcal{A}$.

VT : vertex set of T.
$\mathcal{A}$: alphabet.

A periodic coloring of 3-regular tree



Another periodic coloring of 3-regular tree



$B_n(x) = \{y \in VT \cup ET : d(x,y) \leq n\}$. ($ET$: edge set of T)
$[B_n(x)]$: equivalent class by color-preserving isomorphisms.

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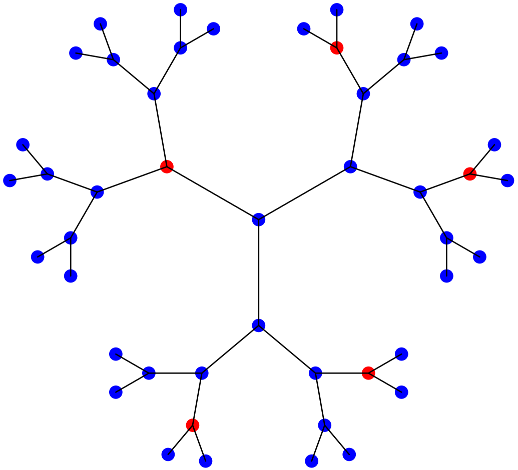
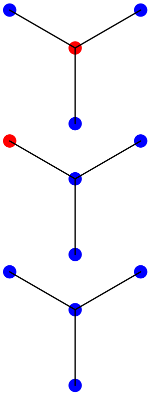
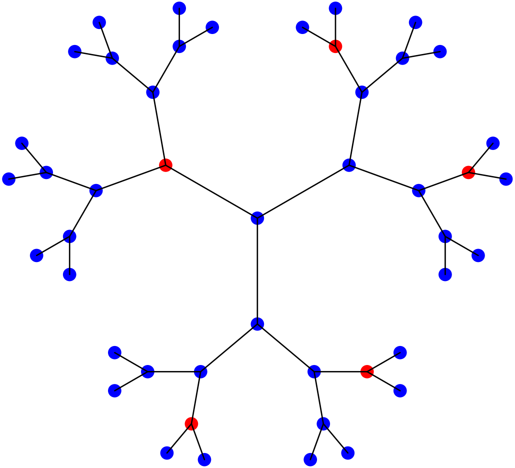
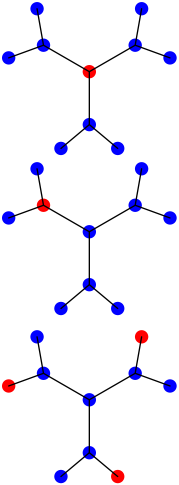




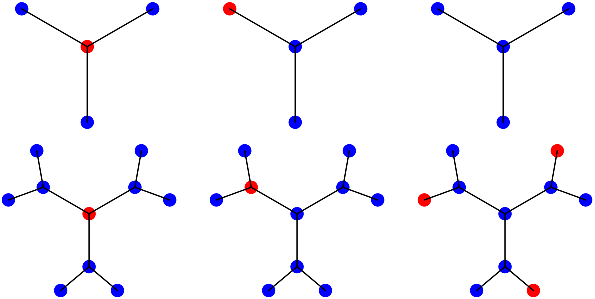
1-balls





2-balls

Finite quotient graph



A B C

A B C

Periodic coloring on the regular tree

$G = \mathrm{Aut}(T)$: the group of automorphisms of T, a locally compact topological group with compact-open topology.

Definition

A coloring $\phi : VT \to \mathcal{A}$ is periodic if there exists a subgroup $\Gamma \subset G$ such that $\Gamma \backslash T$ is a finite graph and ϕ is Γ-invariant, i.e.

$$\phi(\gamma x) = \phi(x), \text{ for all } x \in VT \text{ and } \gamma \in \Gamma.$$

Note that we do not require Γ to be a discrete subgroup of G.

Subball complexity of periodic colorings

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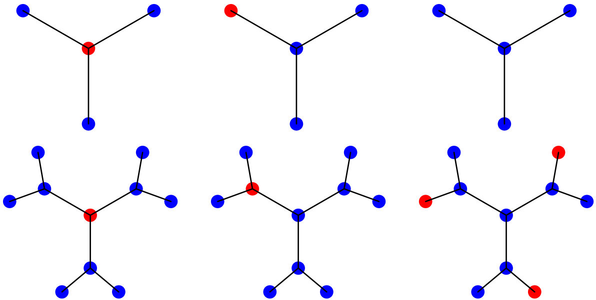


$$b_\phi(1) = 3, \qquad b_\phi(2) = 3$$

Subball complexity of periodic colorings

The subball complexity $b_\phi(n)$ of ϕ is the number of colored n-balls by ϕ.

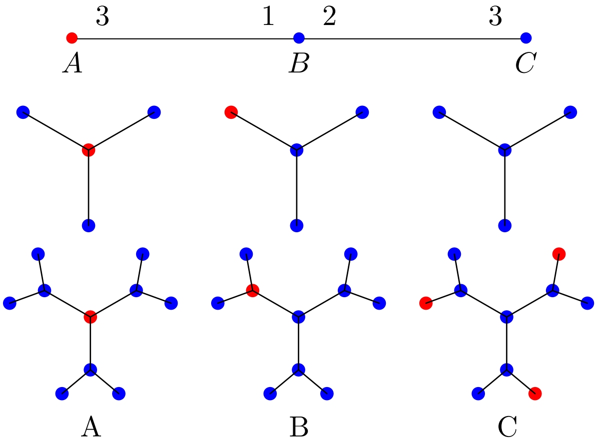
Theorem (K-Lim)

Let $\phi : VT \to \mathcal{A}$ be a coloring. The followings are equivalent.

1. *The coloring ϕ is periodic.*
2. *The subball complexity of ϕ satisfies $b_\phi(n+1) = b_\phi(n)$ for some $n > 0$.*
3. *The subball complexity $b_\phi(n)$ is bounded.*

Note that $\Gamma \backslash T$ has a structure of a graph of groups, i.e., the quotient graph $X = \Gamma \backslash T$ with stabilizers $\Gamma_x = \mathrm{Stab}_\Gamma(x)$ attached to each class $x \in VX \cup EX$.

From $\Gamma \backslash T$ to (T, ϕ) we only need $[\Gamma_x : \Gamma_{\partial(x)}]$.



Motivation

- Discrete subgroup of Lie groups are important in studying Riemannian manifolds of negative curvature.
- M locally symmetric :

$$M \cong \pi_1(M)\backslash\widetilde{M},$$

where $\Gamma = \pi_1(M) \subset G = \mathrm{Isom}^+(\widetilde{M})$ discrete

$$M \cong \Gamma\backslash G/K$$

- Margulis arithmeticity used dichotomy of commensurator group of a discrete subgroup Γ of G :

$$\mathrm{Comm}(\Gamma) = \{g \in G : g\Gamma g^{-1} \cap \Gamma \subset_{\text{f.i.}} \Gamma\}$$

Motivation

- Comparison of Lie groups with automorphism group $\mathrm{Aut}(T)$ of a tree T : study of discrete subgroup of $\mathrm{Aut}(T)$.
- A subgroup Γ is discrete if $\mathrm{Stab}_\Gamma(t)$ are all finite for all $t \in VT, ET$.
- A discrete subgroup Γ is of finite covolume if

$$\mathrm{Vol}(\Gamma\backslash\backslash T) = \sum_{g \in V(\Gamma\backslash T)} \frac{1}{|\mathrm{Stab}_\Gamma(t)|} < \infty.$$

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Q1 Given Γ, is there an invariant of $g \in \mathrm{Aut}(T)$ which distinguishes commensurator elements?

Q2 It there any hierarchy for $g \in G$ with respect to Γ?

Example : Commensurator elements

Let $\Gamma = \langle a_1, \cdots, a_k : a_i^2 = 1 \rangle$, T Cayley graph of Γ, $g \in \mathrm{Aut}(T)$

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For $t \in VT$, a unique $\gamma_t \in \Gamma$ sending the identity to t. Then

$$\gamma_{g(t)}^{-1} \circ g \circ \gamma_t(\mathrm{id}) = \mathrm{id}.$$

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Let $\phi_g(t)$ be the map $\gamma_{g(t)}^{-1} \circ g \circ \gamma_t$ restricted to the 1-sphere of the identity. Then

$$\phi_g : VT \to S_k \text{ is a coloring,}$$

where S_k is the symmetric group.

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Theorem (Lubotzky-Mozes-Zimmer, Avni-Lim-Nevo)

$$g \in Comm(\Gamma) \Leftrightarrow \phi_g \text{ is periodic } \Leftrightarrow b_{\phi_g} \text{ is bounded.}$$

Sturmian colorings

Definition

A coloring ϕ of a k-regular tree T is called Sturmian if

$$b_\phi(n) = n + 2.$$

Note that $b_\phi(n) = n + 2$ is the minimal unbounded subball complexity for non periodic coloring ϕ.

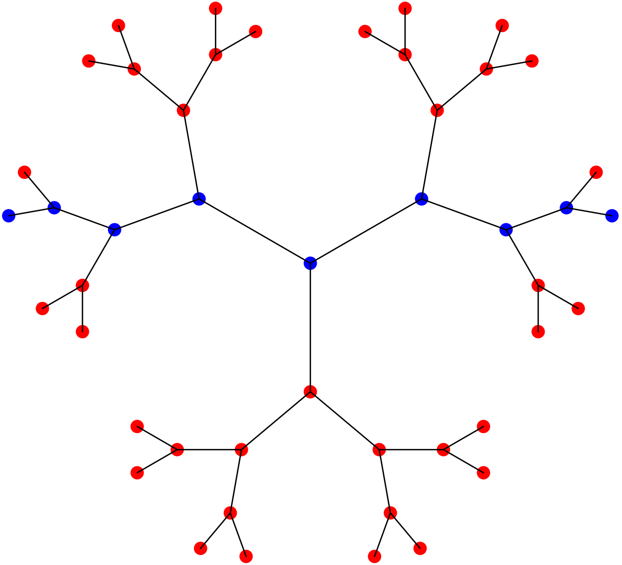
For any coloring ϕ consider the coloring preserving subgroup

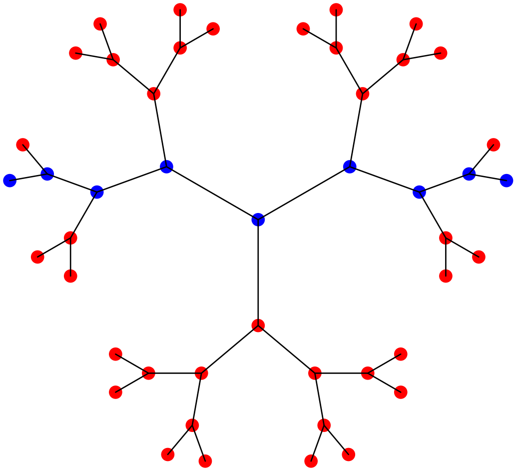
$$\Gamma_\phi = \{g \in \mathrm{Aut}(T) : \phi(gt) = \phi(t)\}.$$

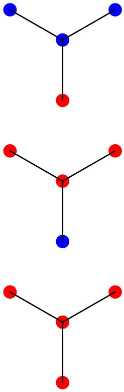
Not if ϕ is not periodic, then the quotient graph $X_\phi = \Gamma_\phi \backslash T$ is an infinite graph.

Question: How does X_ϕ look if ϕ is Sturmian?

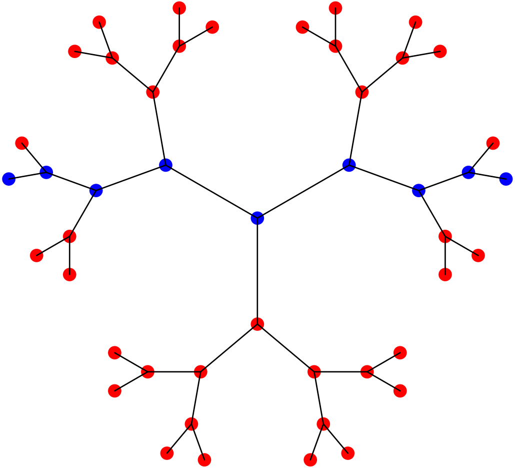
A sturmian coloring of 3-regular tree

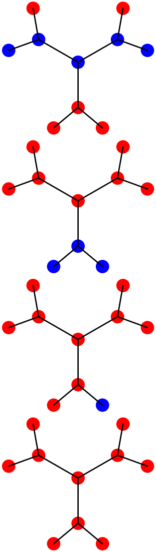




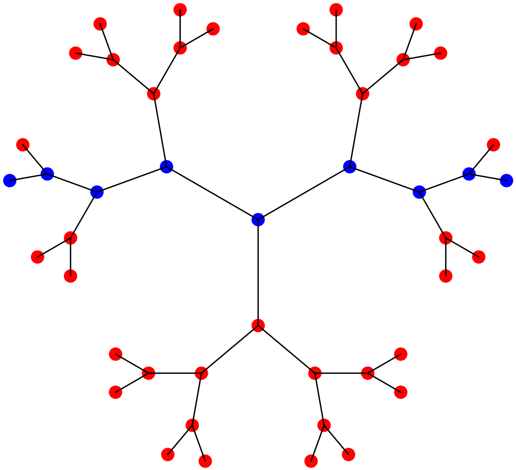


$$b(1) = 3$$

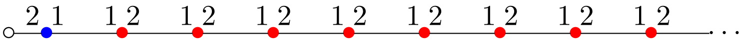




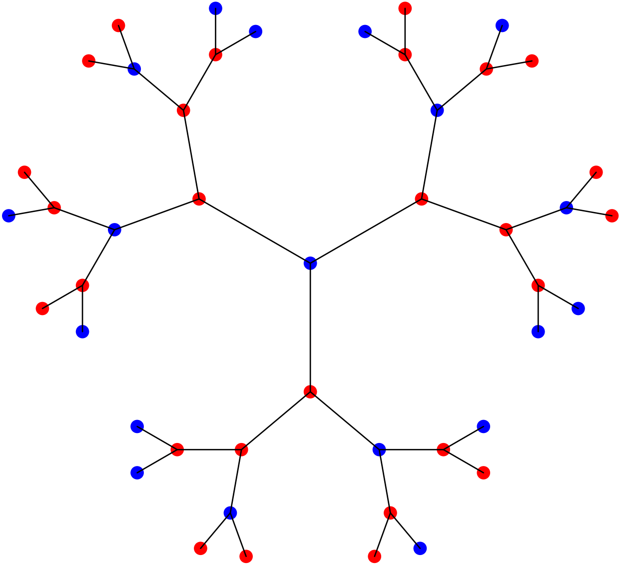
$b(2) = 4$

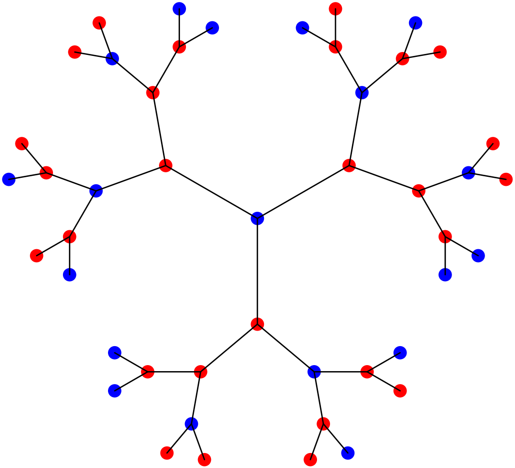


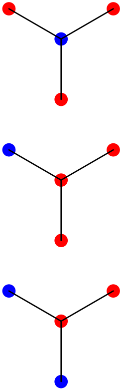
Quotient graph X:



A sturmian coloring of 3-regular tree (bounded type)







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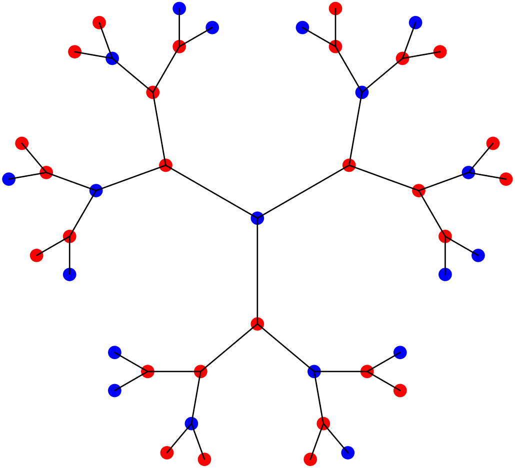

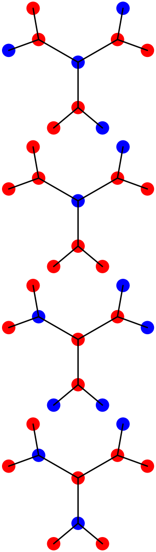

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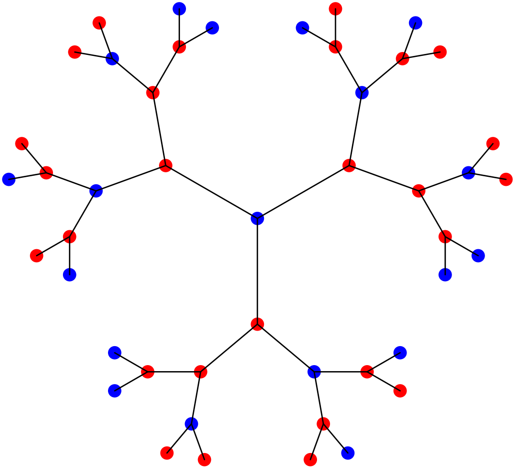

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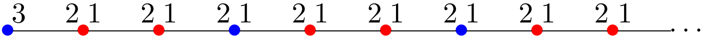
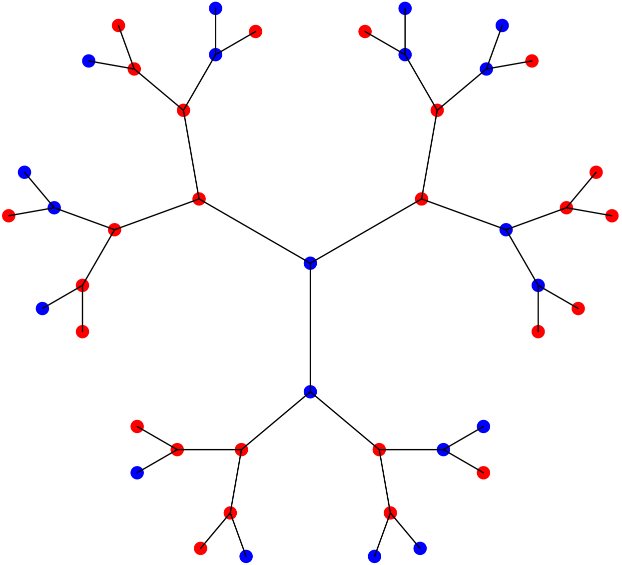
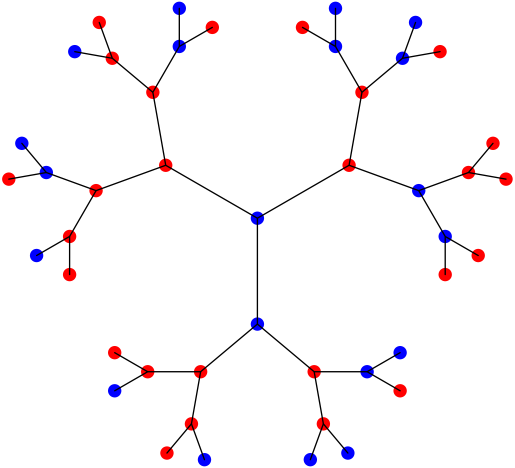
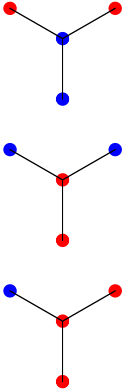


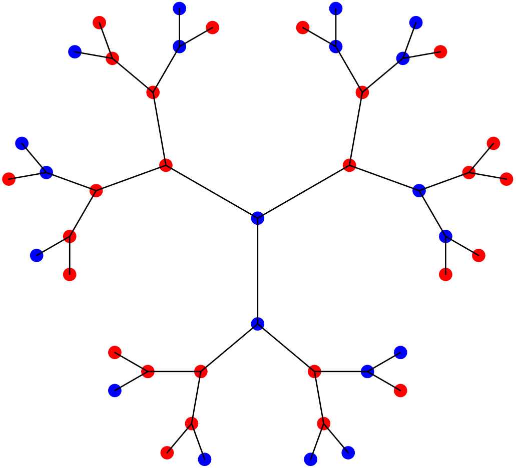
A sturmian coloring of 3-regular tree (unbounded type)

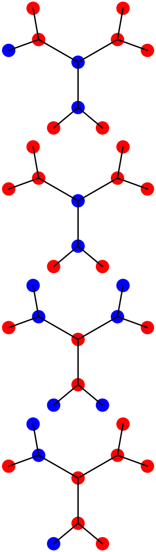






$$b(1) = 3$$





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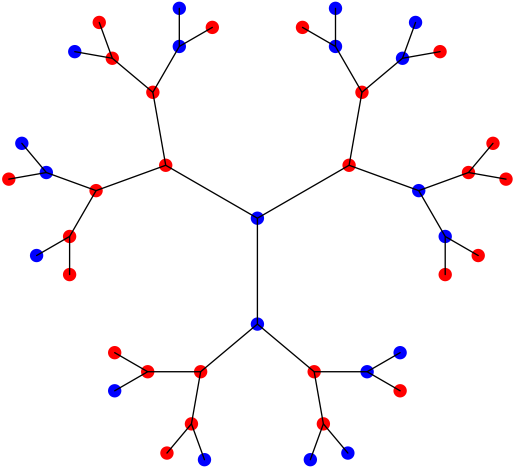

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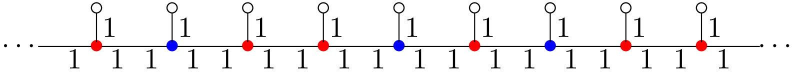
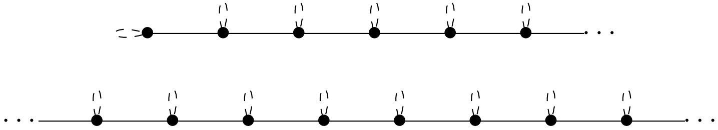


Main Theorem

Let ϕ be a Sturmian coloring of a regular tree T.

Theorem (K-Lim)

There exists a group Γ *acting on* T *such that* ϕ *is* Γ*-invariant, so that* ϕ *is a lifting of a coloring* ϕ_X *on the quotient graph* $X = \Gamma\backslash T$*. The quotient graph* $X = G\backslash T$ *is one of the following two types of graphs.*



Colorings on the 2-regular tree and bi-infinite words

In the 2-regular tree, colored n-balls are $2n+1$-words indentified with revered words.

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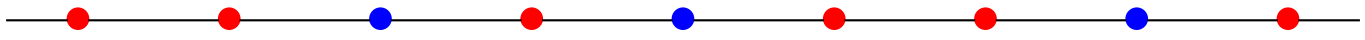
A bi-infinite Sturmian word:

$$\cdots \; a \; b \; \underline{a \; a \; b \; a \; b \; a \; a \; b \; a} \; a \; b \; \cdots$$



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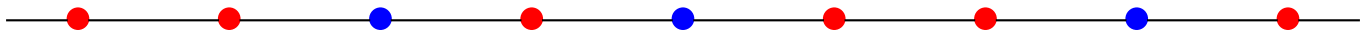
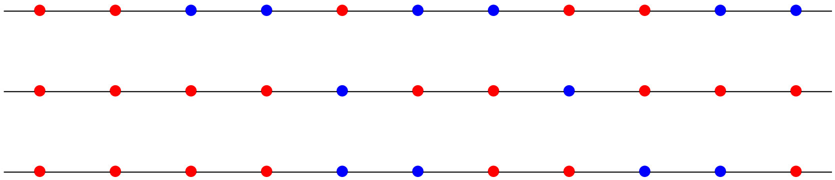
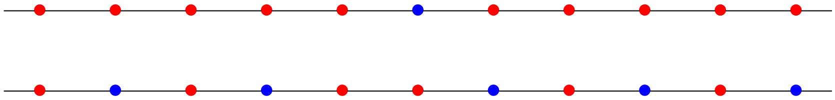


$$\begin{aligned} b(n) &= \frac{1}{2} \cdot \Big(p(2n+1) + \ \#\text{ of palindromic } (2n+1)\text{-word} \Big) \\ &= \frac{1}{2} \cdot \Big((2n+2) + 2 \Big) = n + 2. \end{aligned}$$

Other Sturmian colorings on the 2-regular tree



Also “non-irrational ” colorings:



Sturmian coloring of bounded type

A colored n-ball $[B]$ is special if there exist $x, y \in VT$ such that $[B_n(x)] = [B_n(y)] = [B]$ but $[B_{n+1}(x)] \neq [B_{n+1}(y)]$.

$$\Lambda(x) = \{n \geq 0 : [B_n(x)] \text{ is special}\}.$$

Sturmian coloring of bounded type

A colored n-ball $[B]$ is special if there exist $x, y \in VT$ such that $[B_n(x)] = [B_n(y)] = [B]$ but $[B_{n+1}(x)] \neq [B_{n+1}(y)]$.

$$\Lambda(x) = \{n \geq 0 : [B_n(x)] \text{ is special}\}.$$

A coloring ϕ is of bounded type if $|\Lambda(x)| < \infty, \forall x \in VT$.
Denote $\tau(x) = \max \Lambda(x)$.

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Theorem (K-Lim)

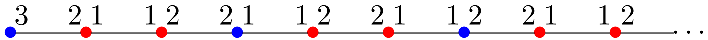
If ϕ is a Sturmian coloring, then there exists a proper infinite quotient graph X of T with

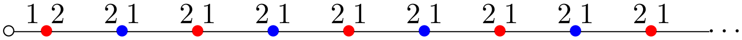
$$\begin{aligned} VX &= \{m, m+1, m+2, \dots, \}, \\ EX &\subset \{[i, i+1], [i+1, i] \mid i \geq m\} \cup \{[i, i] \mid i \geq m\} \end{aligned}$$

and a coloring ϕ_X on X such that $\phi = \phi_X \circ \pi$, where $\pi : T \to X$ is the canonical quotient map and $m = \min\{\tau(x) : x \in VT\}$.

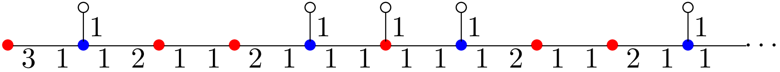
More examples of bounded type Sturmian coloring

Example (Periodic configurations)





Example (Non-periodic edge configuration)



Bounded type Sturmian coloring

Theorem (K-Lim, in preperateion)

Let ϕ be a Sturmian coloring of bounded type on T. Then there exist a periodic coloring $\tilde{\phi}$ on T and an infinite subtree T_0 such that

1. $\tilde{\phi}|_{T_0} = \phi|_{T_0}$.
2. $T = \bigcup_{i=0}^{\infty} f_i(T_0)$ *and* $f_i(T_0) \cap f_j(T_0)$ *contains at most one vertex for* $i \neq j$.
3. $f_j \circ f_i^{-1}$ *is a* ϕ*-preserving automorphism from* $f_i(T_0)$ *to* $f_j(T_0)$ *for all* i, j.

The converse of the theorem does not hold in general. Such a coloring should be a quasi-Sturmian coloring.

Sturmian coloring of unbounded type

Theorem (K-Lim)

If ϕ is a Sturmian coloring of unbounded type, then there exists a proper quotient infinite graph X and a coloring ϕ_X on X such that $\phi = \phi_X \circ \pi$, where π is the projection from the regular tree T to X. Moreover, we have

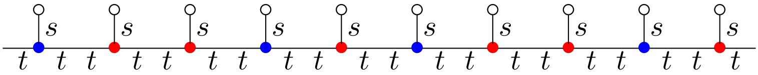
$$VX = \{0, 1, 2, \dots, \}, \quad EX \subset \{[i, i+1] \mid i \geq 0\} \cup \{[i, i] \mid i \geq 0\}$$

or

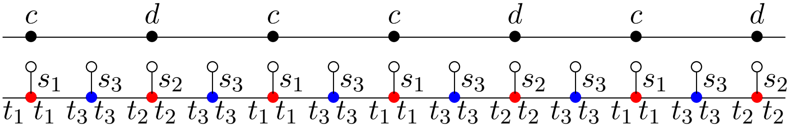
$$\begin{aligned} VX &= \{\dots, -2, -1, 0, 1, 2, \dots, \}, \\ EX &\subset \{[i, i+1] \mid i \in \mathbb{Z}\} \cup \{[i, i] \mid i \in \mathbb{Z}\}. \end{aligned}$$

Examples of unbounded Sturmian colorings

Example (with a periodic edge configuration)



Example (with a periodic vertex configuration)



Example