Lorentzian Geometry I

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Basic Definitions

- $\mathsf{E}^{n,1}$ ($n \geq 2$) is the Lorentzian (flat) affine space with n spatial directions
  - The tangent space: $\mathbf{R}^{n,1}$
  - Choose a point $o \in \mathsf{E}^{n,1}$ as the *origin*
  - Identification of E and its tangent space: $p \leftrightarrow \mathsf{v} = p - o$

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    - Identification of E and its tangent space: $p \leftrightarrow \mathsf{v} = p - o$
- The tangent space $\mathbf{R}^{n,1}$
    - $\mathsf{v} = [v_1, \ldots, v_n, v_{n+1}]^T$
    - The (standard, indefinite) inner product:

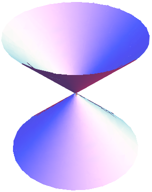
$$\mathsf{v} \cdot \mathsf{w} = v_1 w_1 + \ldots + v_n w_n - v_{n+1} w_{n+1}$$

    - $\mathrm{O}(n,1)$ is the group of matrices which *preserve* the inner product
        - In particular, for any $\mathsf{v}, \mathsf{w} \in \mathbf{R}^{n,1}$ and any $A \in \mathrm{O}(n,1)$

$$A\mathsf{v} \cdot A\mathsf{w} = \mathsf{v} \cdot \mathsf{w}$$

        - $\mathrm{SO}(n,1)$ is the subgroup whose members have determinant 1.
        - $\mathrm{O}^o(n,1) = \mathrm{SO}^o(n,1)$ is the *connected* subgroup containing the identity

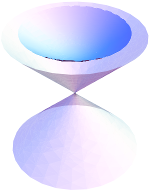
Vectors



- $N = \{v \in \mathbf{R}^{n,1} | v \cdot v = 0\}$ is the *light cone* (or *null cone*) and vectors lying here are called *lightlike*
  - Inside cone: v such that $v \cdot v < 0$, are called *timelike*
  - Outside cone: v such that $v \cdot v > 0$, are called *spacelike*
- Time orientation
  - Choice of nappe, and timelike vectors upper nappe, is a choice of *time orientation*
  - Choose the upper nappe to be the *future*; vectors on or inside the upper nappe are *future pointing*

Models of Hyperbolic Spaces

- One sheet of hyperboloid
  - $\mathrm{H}^n \cong \{\mathrm{v} \in \mathbf{R}^{n,1} | \mathrm{v} \cdot \mathrm{v} = -1$, and future pointing$\}$
  - $\mathrm{w} \cdot \mathrm{w} > 0$ for w tangent to hyperbola.
  - Defined metric has constant curvature -1.
  - Geodesics $= \{$Planes thru $o\} \cap \{$hyperboloid$\}$
- Projective model
  - $\mathrm{v} \sim \mathrm{w}$ if $\mathrm{v} = k\mathrm{w}$ for $k \neq 0$, written $(\mathrm{v}) = (\mathrm{w})$
  - $\mathrm{H}^n \cong \{\mathrm{v} \in \mathbf{R}^{n,1} | \mathrm{v} \cdot \mathrm{v} < 0\} / \sim$
  - Homogeneous coordinates $(\mathrm{v}) = [v_1 : v_2 : \ldots : v_n]$
- Klein model
  - Project onto $v_n = 1$ plane.
  - Geodesics are straight lines.
  - Not conformal.



Isometries

- Linear Isometries
  - $O(n,1)$ has four connnected components.
  - Isometries of H^n
- Affine isometries: $\mathcal{A} = (A, a) \in \mathrm{Isom}(E)$
  - $A \in O(n,1)$ and $a \in \mathbf{R}^{n,1}$
  - $\mathcal{A}(x) = A(x) + a$

Proposition

For any affine isometry, $x \mapsto A(x) + a$, if A does not have 1 as an eigenvalue, then the map has a fixed point.

Proof.

If A does not have 1 has an eigenvalue, you can always solve $A(x) + a = x$, or $(A - I)(x) = -a$ □

Three dimensions

- More on products
  - $v^{\perp} = \{w | w \cdot v = 0\}$
    - If v is spacelike, $v^{\perp}$ defines a geodesic.
    - If v is lightlike, $v^{\perp}$ is tangent to lightcone at v.
  - *(Lorentzian) cross product*
    - $v \times w$ is (Lorentzian) orthogonal to v and w.
    - Defined by $v \cdot (w \times u) = \mathrm{Det}(v, w, u)$.

- Upper half plane model of the hyperbolic plane
  - $U = \{z \in \mathbf{C} | \mathrm{Im}(z) > 0\}$ with boundary $\mathbf{R} \cup \{\infty\}$.
  - Geodesics are arcs of circles centered on $\mathbf{R}$ or vertical rays.
  - $\mathrm{Isom}^{+}(\mathrm{H}^2) \cong \mathrm{PSL}(2, \mathbf{R})$

$A \in \mathsf{SO}^o(2,1)$

- All A have 1 eigenvalue.
- Classification: Nonidentity A is said to be ...
  - *elliptic* if it has complex eigenvalues.
    - The 2 complex eigenvectors are conjugate.
    - The fixed eigenvector A^0 is timelike.
    - Acts like rotation about fixed axis.
  - *parabolic* if 1 is the only eigenvalue.
    - The fixed eigenvector A^0 is *lightlike*.
    - On H^2, fixed point on boundary and orbits are *horocycles*
  - *hyperbolic* if it has 3 distinct real eigenvalues $\lambda < 1 < \lambda^{-1}$
    - Fixed eigenvector A^0 is spacelike.
    - The *contracting* eigenvector A^- and *expanding* eigenvector A^+ are lightlike.
    - $A^0 \cdot A^{\pm} = 0$
- $\mathcal{A}(x) = A(x) + \mathsf{a}$ is called *elliptic /paraobolic/ hyperbolic* if A is elliptice /parabolic/ hyperbolic.

Hyperbolic affine transformations

- More on linear part
  - Choose $A^{\pm}$ are future pointing and have Euclidean length 1.
  - Choose so that $A^0 \cdot (A^- \times A^+) > 0$ and $A^0 \cdot A^0 = 1$.
  - $(A^0)^\perp$ determines the *axis* of A on the hyperbolic plane.
- *The Margulis invariant* for a hyperbolic $\mathcal{A} = (A, \mathrm{a})$
  - There exist a unique invariant line $C_{\mathcal{A}}$ parallel to A^0.
  - The *Margulis invariant*: for any $x \in C_{\mathcal{A}}$

$$\alpha(\mathcal{A}) = (\mathcal{A}(x) - x) \cdot A^0$$

    - Signed Lorentzian length of unique closed geo in $\mathsf{E}^{2,1}/\langle \mathcal{A} \rangle$.
    - $\alpha(\mathcal{A}) = 0$ iff $\mathcal{A}$ has a fixed point.
    - Invariant given choice of $x \in \mathsf{E}$.
    - Invariant under conjugation (α is a *class function*), and determines conjugation class for a fixed linear part.
    - $\alpha(\mathcal{A}^n) = |n|\, \alpha(\mathcal{A})$

Proper actions

- For any discrete G action on a locally compact Hausdorff X, if G is *proper* then X/G is Hausdorff.
  - Alternatively, G is to act freely *properly discontinuously* on X.
  - (Bieberbach) For $X = \mathbf{R}^n$ and discrete $G \subset \mathrm{Isom}(X)$, if G acts properly on X then G has a finite index subgroup $\cong \mathbb{Z}^m$ for $m <= n$.
- Cocompact affine actions

Conjecture (Auslander)

For $X = \mathbf{R}^n$ and discrete $G \subset \mathrm{Aff}(\mathbf{R}^n)$, if G acts properly and cocompactly on X then G is virtually solvable.

- No free groups of rank $>= 2$ in virtually solvable gps.
- True up to dimension 6.
- (Milnor) Is Auslander Conj. true if "cocompact" is removed? NO.

Margulis Opposite Sign Lemma

Lemma (Margulis' Opposite Sign)

If $\alpha(\mathcal{A})$ *and* $\alpha(\mathcal{B})$ *have opposite signs then* $\langle \mathcal{A}, \mathcal{B} \rangle$ *does not act properly on* $\mathsf{E}^{2,1}$.

- The signs for elements of proper actions must be the same.
- Opposite Sign Lemma true in $\mathsf{E}^{n,n-1}$
  - When n is odd, $\alpha(\mathcal{A}^{-1}) = -\alpha(\mathcal{A})$, so no groups with free groups (rank ≥ 2) act properly.
  - Can find counterexamples to "noncompact Auslander" in $\mathsf{E}^{2,1}, \mathsf{E}^{4,3}, \ldots$.

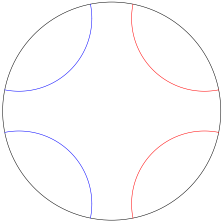
Margulis space-times

- First examples

Theorem (Margulis)

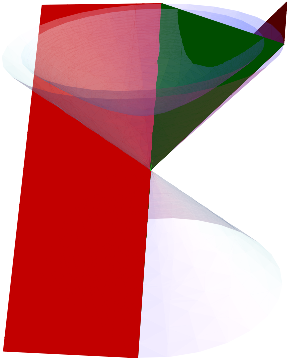
There exist discrete free groups of $\mathrm{Aff}(\mathsf{E}^{2,1})$ *that act properly on* $\mathsf{E}^{2,1}$.

- Next examples
  - Free discrete groups in $\langle A_1, A_2, ..., A_n\rangle \subset \mathrm{Isom}(\mathsf{H}^2)$.
  - Domain bounded by $2n$ nonintersecting geodesics $\ell_n^{\pm}$ such that $A_i(\ell_i^-) = \ell_i^+$.



Crooked Planes

- Problem: extend notion of lines in H^2 to $E^{2,1}$.
- A *Crooked Plane*
  - *Stem* is perpendicular to spacelike vector v through *vertex* p inside the lightcone at p.
    - *Spine* is the line through p and parallel to v
  - *Wings* are half planes tangent to light cones at boundaries of stem, called the *hinges*.
- A *Crooked half-space* is one of the two regions in $E^{2,1}$ bounded by a crooked plane.



Crooked domains

Theorem (D)

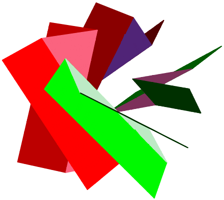
Given discrete $\Gamma = \langle \mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n \rangle \subset \mathrm{Isom}(\mathsf{E}^{2,1})$*. If there exist* $2n$ *mutually disjoint crooked half spaces* $\mathcal{H}_n^{\pm}$ *such that* $\mathcal{A}_i(\mathcal{H}_i^-) = \mathsf{E}^{2,1} \setminus \mathcal{H}_i^+$*, then* Γ *is proper.*

- Example of a “ping-pong” theorem.
- Finding proper actions
  - Start with a free discrete linear group.
  - Find *disjoint* halfspaces whose complement is domain for a linear part.
  - *Separate* half planes, giving rise to proper affine group.

Crooked domains

- Two pair of *disjoint* halfspaces at the origin.

- Separated



Results

Theorem (D)

Given every free discrete group $G \subset \mathrm{SO}(2,1)$ *there exists a proper subgroup* $\Gamma \subset \mathrm{Isom}(\mathsf{E}^{2,1})$ *whose underlying linear group is* G.

Theorem (Danciger- Guéritaud - Kassel)

For every discrete $\Gamma \subset \mathrm{Isom}(\mathsf{E}^{2,1})$ *acting properly on* $\mathsf{E}^{2,1}$, *there exists a crooked fundamental domain for the action.*

References

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- (with V. Charette) *Complete Lorentz 3-manifolds*, Cont. Math. 630, (2015), pp. 43 72