Limit sets of discrete groups

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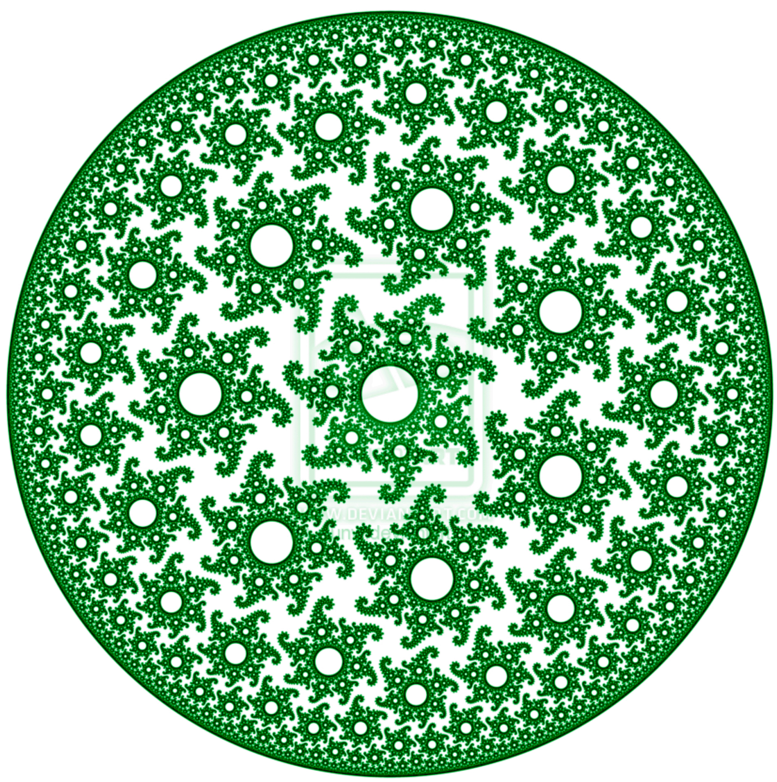

INTRODUCTION

The limit set of a discrete group of transformations is a key-concept in dynamics as well as in various other areas of mathematics. There is not one definition of limit set that is universal and applies in all settings, there are rather several such concepts, defined in different settings, having various properties and characteristics. In a naive way we can say when we have an infinite family $\mathcal{F}$ of transformations in some space X, limit sets help us to understand the long term behavior of the family $\mathcal{F}$. Philosophically, the limit set ought to be a closed subspace of X which is invariant under the given family of transformations, and it is there where the dynamics concentrates. The action on the complement, which is an open invariant set, should be “mild” in some sense.

Limit sets spring naturally from various sources. For instance, if X is a topological space and $X \xrightarrow{f} X$ a continuous function then the iterates of f,

$$f_1 := f \ , \ f_2 := f \circ f_1 \ , \ f_n := f \circ f_{n-1} \ , \cdots$$

form a semigroup $\{f_n\}_{n\in\mathbb{N}}$ and the ω-limit of f is the set of accumulation points of the orbits $\{f_n(x)\}_{n\in\mathbb{N}}$, $x \in X$. If f is a homeomorphism, so there is a continuous inverse function f^{-1}, then setting f_0 to be the identity, f actually generates a group $\{f_n\}_{n\in\mathbb{Z}}$ where f_{-n} is by definition the n^{th} iterate of the inverse f^{-1}, *i.e.*

$$f_{-1} := f^{-1} \ , \quad f_{-2} = f^{-1} \circ f^{-1} \ \cdots$$

Then one also has the concept of the α-limit of f: The set of accumulation points of the backwards orbits $\{f_{-n}(x)\}_{n\in\mathbb{N}}$, $x \in X$. These concepts play a key-role in the classical theory of dynamical systems developed by S. Smale and others.

If $X \xrightarrow{f} X$ is continuous but is not a homeomorphism, then there is not a continuous inverse f^{-1}, but we can still look at the inverse image of points: $f^{-1}(x)$ are the points $y \in X$ such that $f(y) = x$. Then we have the backwards orbits $\{f_{-n}(x)\}_{n\in\mathbb{N}}$, $x \in X$, and we can look at the set of points where these orbits accumulate. For instance when X is the Riemann sphere $\mathbb{S}^2$ and f is a rational function of degree ≥ 2, the set of accumulation points of the backwards orbits (of all but some points) form the Julia set, and we all know that the study of these sets is in itself a rich and fascinating subject.

In these lectures I will study first limit sets of discrete groups of transformations of the complex projective line $\mathbb{CP}^1$, a topic which springs from the classical work of Poincaré for (classical) Kleinian groups: These are groups generated by Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ with a, b, c, d complex numbers such that $ad - bc \neq 0$ (see section 1 below for more on the subject). The set of all such Möbis transformations forms a Lie group of real dimension 6, isomorphic to $\mathrm{PSL}(2, \mathbb{C})$, the group of holomorphic automorphisms of the complex projective line $\mathbb{CP}^1$, which is biholomorphic to the Riemann sphere $\mathbb{S}^2$.

A subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is said to be discrete if it is discrete as a topological subspace, *i.e.*, each point in it has a neighborhood that contains no other points in the subgroup.

Then the limit set of such a group is defined to be the set of accumulation points of the orbits. This set has very rich dynamics and a fascinating geometry.

The study of limit sets of Kleinian groups, and their relation with the geometry and topology of Riemann surfaces and hyperbolic 3-manifolds has been for decades, and continues to be, one of the most active areas of research in mathematics.

The question we shall explore in these lectures is how this concept of limit set generalizes to higher dimensions, a topic which still is in its childhood though it is starting to be understood, particularly in dimension two thanks to the work of various people. Explicit references are given along the text.

These notes are arranged into three sections, one for each lecture.

Section 1 is about the classical theory of Kleinian groups acting on the 1-dimensional complex projective space. Our aim here is to give a quick introduction to this rich and fascinating subject, paving the ground for the following sections, and no doubt for several other lectures during this meeting. We define what the limit set is in this setting, and we discuss some basic properties of the limit set which serve as guidelines for following sections, including its relation with the region where the action is properly discontinuous and equicontinuous, two important concepts that play key roles in the subject.

In Section 2 we look at discrete groups of transformations of complex projective spaces in higher dimensions and notice that in this setting, the classical definition of the limit set is not "good enough". We discuss several interesting families of groups and limit sets, including the Kulkarni limit set, which in dimension 1 coincides with the usual one and plays a key role in dimension 2.

In section 3 we focus on complex dimension 2 and describe how in this dimension, the Kulkarni limit set has remarkable similarities with the classical limit set in dimension 1. These are summarized in a table at the end of these notes.

The material in these notes is largely due to Angel Cano, Waldemar Barrera, Juan Pablo Navarrete and Alberto Verjovsky; we refer to the bibliography for more on the subject. This all shows, I believe, that in complex dimension two the Kulkarni limit set is "the good concept" of limit set. Yet, when we come into higher dimensions, this is not so clear. For instance, it is proved in [1] that the limit set for Schottky groups in $\mathbb{CP}^3$ introduced in [36] and discussed in Section 2 of these notes, is smaller than and contained in the Kulkarni Limit set, and it is a closed invariant set such that the action on its complement is properly discontinuous. That seems related with [24], where the authors use the Cartan's decomposition theorem to give a rather subtle and interesting definition of a limit set which is in some sense "finer" than the Kulkarni limit set.

I am very grateful to my aforementioned colleagues in Mexico: Cano, Barrera, Navarrete and Verjovsky, as well as to John Parker, Nikolay Gusevskii, Adolfo Guillot and Carlos Cabrero, for the fruitful collaboration we have had and for very many useful discussions that have led to what I will talk about in these lectures. I am also grateful to the ICTP for allowing us to have this meeting, and to CONACYT (Mexico) for its financial support through various grants.

Contents

1 INTRODUCTION TO KLEINIAN GROUPS

1.1 Group actions

Let G be a group and M a smooth manifold. An *action* of G on M means a function:

$$\Phi : G \times M \longrightarrow M$$

which preserves the group structure of G. That is, Φ satisfies:

i) If e denotes the identity in G, then $\Phi(e, x) = x$ for all $x \in M$.

ii) If $\cdot$ denotes the product in G, then for all $x \in M$ and $g, h \in G$ we have

$$\Phi(g \cdot h, x) = \Phi(g, \Phi(h, x))\,.$$

Intuitively, an action means a method for "multiplying" elements of the group G by points in M, so that the result is a point in M. Notice that in the definition above we are actually "multiplying " the elements of G by the points in M by the left, so what we have is a *left action*. One has the equivalent notion for right actions. Yet, for the sake of simplicity, in these notes we will speak just of "actions", not specifying in general whether they are right or left actions.

We assume in the sequel that G is actually a Lie group. This means that G is itself a smooth manifold and the group operations in G,

$$g \mapsto g^{-1} \qquad \text{and} \qquad (g,h) \mapsto g \cdot h\,,$$

are differentiable. We assume further that the map Φ is differentiable. Notice that in this situation, every subgroup H of G acts on G by multiplication (on the right or on the left). Also, if G acts on a manifold M, then the restriction of the action to H gives an H-action on M. Notice that G and M may or may not be compact, neither we are saying anything about their dimensions: In fact we will be most interested in the case when H is a Lie subgroup of dimension 0 of some Lie group.

Observe that for each $g \in G$ we have a smooth map $\phi_g : M \to M$ given by $\phi_g(x) = \Phi(g,x)$. This map has an inverse given by $x \mapsto \Phi(g^{-1},x)$. Hence each ϕ_g is a diffeomorphism of M. That is, the differentiable group action Φ can be regarded as a family of diffeomorphisms of M parameterized by G.

Similarly, for each $x \in M$ we have a smooth map $\mathcal{O}_x : G \to M$ defined by $g \mapsto \Phi(g,x)$. The image $G(x)$ of $\mathcal{O}_x$ is called *the orbit* of x under the action of G:

$$G(x) = \{y \in M \mid y = \Phi(g,x) \text{ for some } g \in G\}\,.$$

Given a G-action Φ on M, for each $x \in M$ one has *the stabilizer* of x, also called *the isotropy* subgroup of x, defined by:

$$G_x = \{g \in G \mid \Phi(g,x) = x\}\,.$$

That is, G_x consists of all the elements in G that leave the point x fixed.

An action is called *free* if all stabilizers are trivial, *i.e.*, if for all $x \in M$ and all $g \in G \setminus e$ we have $\Phi(g,x) \neq x$.

For simplicity, if a group G acts on M we denote the action of $g \in G$ at each point by $g \cdot x$.

Examples 1.1 i. Given fixed real numbers λ_1, λ_2, we may let $\mathbb{R}$ act on $\mathbb{R}^2$ by

$$t \cdot (x_1,x_2) \mapsto (e^{\lambda_1 t}x_1, e^{\lambda_1 t}x_2)\,.$$

An action of the real numbers $\mathbb{R}$ on a manifold M is called *a flow* or also *a one parameter group* of diffeomorphisms of M.

ii. Let $O(2)$ be *the orthogonal group* generated by reflections on all lines through the origin in $\mathbb{R}^2$, and let $Aff(2,\mathbb{R})$ be the group generated by reflections on all lines in $\mathbb{R}^2$. These groups act on $\mathbb{R}^2$ in the obvious way. Now, given integers $p,q,r \geq 2$ such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1\,,$$

let $T := T_{p,q,r}$ be a triangle in $\mathbb{R}^2$ with inner angles $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r}$, [We leave as an exercise to show that the only possible triples are, up to permutation, the triples $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$] and let $\Sigma_{p,q,r}$ be the subgroup of $Aff(2, \mathbb{R})$ generated by the reflections on the edges of T. Then $\Sigma_{p,q,r}$ acts on $\mathbb{R}^2$ in such a way that the various (infinitely many) copies of T cover the plane.

Notice that if ℓ_1 and ℓ_2 are the lines determined by two edges of T that determine, say, the angle π/p, then the reflection on ℓ_1 followed by the reflection on ell_2 is a rotation by an angle $2\,\pi/p$ around the point where these two lines meet.

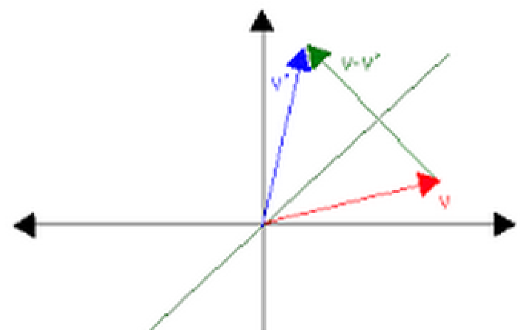

Figure 1: Reflection on a line through the origin.

Hence the isotropy of this point, which is a vertex of T, is a cyclic group of order $2p$.

iii. Let $\mathrm{O}(n)$ be, more generally, the group of linear maps of $\mathbb{R}^n$ generated by the reflections on hyperplanes through the origin, and let SO(n) be the index two subgroup of $\mathrm{O}(n)$ consisting of elements that can be expressed by an even number of reflections. This is the group of rotations of $\mathbb{R}^n$. Both of these groups preserve the usual metric in $\mathbb{R}^n$, so they leave invariant each sphere centred at the origin, and we may think of each of them as acting on the unit sphere. It is clear that the origin in $\mathbb{R}^n$ is a fixed point for the corresponding actions, that is $g \cdot \underline{0} = \underline{0}$ for all $g \in \mathrm{O}(n)$. We leave it as an exercise to show that all other points have isotropy $\mathrm{O}(n-1)$ (and SO(n − 1)).

iv. Let f be a diffeomorphism of a manifold M. For instance, let M be the 2-sphere $\mathbb{S}^2$ and identify it with the extended plane $\widehat{\mathbb{R}}^2 := \mathbb{R}^2 \cup \infty$ by stereographic projection, so that the origin $(0, 0)$ corresponds to the South pole S while ∞ corresponds to the North pole. And let f be the map in $\widehat{\mathbb{R}}^2$ defined by $(x, y) \mapsto (\frac{1}{2}x, 2y)$. Now *iterate* this function. That is, look at the family of maps $f_1 := f$, $f_2 = f \circ f_1$, $f_3 = f \circ f_2$ and so on. Define also $f_0 := Id$ and set $f_{-1} := f^{-1}$; we may thus iterate f also *backwards* and get a family of maps $\{f_n\}_{n \in \mathbb{Z}}$. Then the assignment $n \mapsto f_n$ determines an action of the integers $\mathbb{Z}$ in $\mathbb{S}^2$.

Notice that in this case we have two points which are fixed by the action, the poles S and N. The points in the axes converge to S and N (either when travelling forwards or backwards), while all other points converge to N.

Observe that in this example we can replace f by any other diffeomorphism and get an action of $\mathbb{Z}$ on $\mathbb{S}^2$. This will be relevant in the sequel. Notice too that if we replace f by some other function which is not a diffeomorphism, then we do not have an inverse f^{-1}. In this case we do not have an action of $\mathbb{Z}$. Yet, we can iterate f forwards and look at the forwards orbits of points. And we can also look at the inverse images of points, and get *the backwards orbit.*

1.2 Inversions and the Möbius group

General references for this and the following sections in this chapter are Beardon's book [4] and the excellent notes of M. Kapovich [25].

Let us consider now another type of transformations, which are analogous to reflections, the inversions. Given a circle $C = C(a, r)$ in the plane $\mathbb{R}^2$ with centre at a point $a \in \mathbb{R}^2$ and radius r, the inversion in C is the map $\iota = \iota(a, r)$ of the 2-sphere $\mathbb{S}^2 \cong \widehat{\mathbb{R}}^2 := \mathbb{R}^2 \cup \infty$ defined for each $z = (x, y) \neq a, \infty$ by:

$$\iota_{a,r}(x, y) = (a_1, a_2) + \frac{r^2}{|(x, y) - (a_1, a_2)|^2} \big(x - a_1, y - a_2\big) ;$$

define $\iota(a) = \infty$ and $\iota(\infty) = a$. Notice that given the circle C, say with centre 0 and radius $r > 0$, the inversion in C carries a point P to unique point in the ray $\overrightarrow{0, P}$ such that $|P|\,|P'| = r^2$. This definition obviously extends to $(n-1)$-spheres in $\mathbb{S}^n \cong \mathbb{R}^n \cup \infty$.

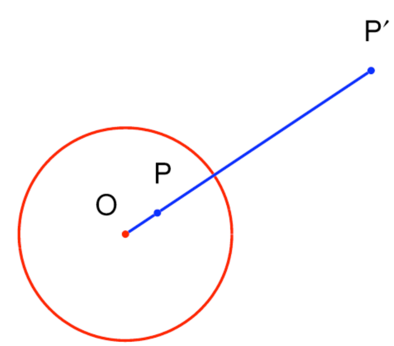

Figure 2: Inversion on a circle in the plane

We remark that for circles of maximal length (*i.e.*, radius 1 in the 2-sphere) this map is just a reflection in the corresponding line in $\mathbb{R}^2$.

It is an exercise to show that inversions are conformal maps, *i.e.*, they preserve angles. That is, if two curves in $\mathbb{S}^2$ meet with an angle θ, then their images under an inversion also meet with an angle θ. Moreover, one has that if C_1 , C_2 are circles in $\mathbb{S}^2$ and ι_1 is the inversion with respect to C_1, then $\iota_1(C_2) = C_2$ if and only if C_1 and C_2 meet orthogonally. We leave the prove as an exercise (Show first that two circles C_1, C_2 in $\mathbb{S}^2$ meet orthogonally at the points P_1 y P_2 if and only if the centre z_2 of C_2 is the meeting point of the lines $\mathcal{L}_\infty$ and $\mathcal{L}'_\infty$, which are tangent to C_1 at P_1 and P_2, and conversely, the centre z_1 of C_1 is the meeting point of the lines $\mathcal{L}_\in$ and $\mathcal{L}'_\in$, tangent to C_2 at these points.)

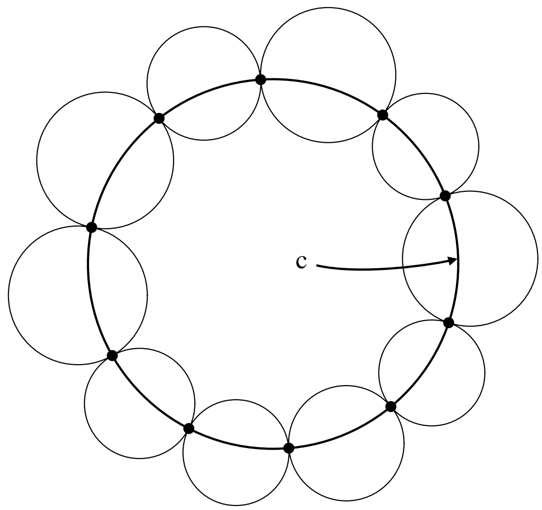

Figure 3: A collar of circles having C as a common orthogonal circle.

In fact the same statement holds in all dimensions (with essentially the same proof):

Theorem 1.2 *Let C_1^{n-1} , C_2^{n-1} be spheres of dimension $n-1$ in $\mathbb{S}^n$ and ι_1 the inversion with respect to C_1. Then $\iota_1(C_2) = C_2$ if and only if C_1 and C_2 meet orthogonally.*

We now let $\text{Möb}(\mathbb{S}^n)$ be the group of diffeomorphisms of $\mathbb{S}^n \cong \widehat{\mathbb{R}} = \mathbb{R}^n \cup \{\infty\}$ generated by inversions on all $(n-1)$-spheres in $\mathbb{S}^n$, and let $\text{Möb}(\mathbb{B}^n)$ be the subgroup of $\text{Möb}(\mathbb{S}^n)$ consisting of maps that preserve the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$.

Notice that if the $(n-1)$-sphere $\mathcal{S}_1$ meets $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ orthogonally then $\mathcal{C} := \mathcal{S}_1 \cap \mathbb{S}^{n-1}$ is an $(n-2)$-sphere in $\mathbb{S}^{n-1}$ and the restriction to $\mathbb{S}^{n-1}$ of the inversion $\iota_{\mathcal{S}_1}$ coincides with the inversion on $\mathbb{S}^{n-1}$ defined by the $(n-2)$-sphere $\mathcal{C}$. In other words one has a canonical group homomorphism $\text{Möb}(\mathbb{B}^n) \to \text{Möb}(\mathbb{S}^{n-1})$.

Conversely, given an $(n-2)$-sphere $\mathcal{C}$ in $\mathbb{S}^{n-1}$ there is a unique $(n-1)$-sphere $\mathcal{S}$ in $\mathbb{S}^n$ that meets $\mathbb{S}^{n-1}$ orthogonally at $\mathcal{C}$. The inversion

$$\iota_{\mathcal{C}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$$

extends canonically to the inversion:

$$\iota_{\mathcal{S}} : \mathbb{B}^n \to \mathbb{B}^n \,,$$

thus giving a canonical group homomorphism $\text{Möb}(\mathbb{S}^{n-1}) \to \text{Möb}(\mathbb{B}^n)$, which is obviously the inverse morphism of the previous one. Thus one has:

Theorem 1.3 *There is a canonical group isomorphism* $\textit{Möb}(\mathbb{B}^{\rm n}) \cong \textit{Möb}(\mathbb{S}^{\rm n-1})$.

Definition 1.4 We call $\text{Möb}(\mathbb{B}^n)$ (and also $\text{Möb}(\mathbb{S}^n)$) *the general Möbius group* of the ball (or of the sphere).

The subgroup $\text{Möb}_+(\mathbb{B}^n)$ of $\text{Möb}(\mathbb{B}^n)$ of words of even length consists of the elements in $\text{Möb}(\mathbb{B}^n)$ that preserve the orientation. This is an index two subgroup of $\text{Möb}(\mathbb{B}^n)$. Similar considerations apply to $\text{Möb}(\mathbb{S}^n)$. We call $\text{Möb}_+(\mathbb{B}^n)$ and $\text{Möb}_+(\mathbb{S}^n)$ *Möbius groups* (of the ball and of the sphere, respectively).

It is easy to see that $\text{Möb}(\mathbb{S}^n)$ includes:

- Euclidean translations: $t(x) = x+a$, where $a \in \mathbb{R}^n$. These are obtained by reflections on parallel hyperplanes.
- Rotations: $t(x) = Ox$, where $O \in \text{SO(n)}$; obtained by reflections on hyperplanes through the origin.
- Homotecies, obtained by inversions on spheres with same centre and different radius.

In fact one has:

Theorem 1.5 *The group* $\textit{Möb}(\mathbb{S}^n)$ *of Möbius transformations is generated by the previous transformations: Translations, rotations and homotecies, together with the inversion:* $t(x) = x/\|x\|^2$.

It is clear that the rotations are actually contained in $\text{Möb}_+(\mathbb{B}^n)$, since hyperplanes through the origin meet transversally the unit sphere in R^n. In fact one has that $\text{Möb}_+(\mathbb{B}^n)$ contains the orthogonal group SO(n) as the stabilizer (or isotropy) subgroup at the origin 0 of its action on the open ball $\mathbb{B}^n$. The stabilizer of 0 under the action of the full group $\text{Möb}(\mathbb{B}^n)$ is $O(n)$. This implies that $\text{Möb}_+(\mathbb{B}^n)$ acts transitively on the space of lines through the origin in $\mathbb{B}^n$. Moreover, $\text{Möb}_+(\mathbb{B}^n)$ clearly acts also transitively on the intersection with $\mathbb{B}^n$ of each ray through the origin. Thus it follows that $\text{Möb}_+(\mathbb{B}^n)$ acts transitively on $\mathbb{B}^n$. In other words we have:

Theorem 1.6 *The group* $\textit{Möb}_+(\mathbb{B}^n)$ *acts transitively on the unit open ball* $\mathbb{B}^n$ *with isotropy* SO(n). *Furthermore, this action extends to the boundary* $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ *and defines a canonical isomorphism between this group and the Möbius group* $\textit{Möb}_+(\mathbb{S}^{n-1})$.

We remark that for $n > 2$, $\text{Möb}_+(\mathbb{S}^{n-1})$ is the group of (orientation preserving) conformal automorphisms of the sphere (see for instance Apanasov's book). That is, we have:

Theorem 1.7 *For all $n > 2$ we have group isomorphisms*

$$M\ddot{o}b_+(\mathbb{B}^n) \cong M\ddot{o}b_+(\mathbb{S}^{n-1}) \cong \mathrm{Conf}_+(\mathbb{S}^{n-1}) .$$

In fact the previous constructions show that every element in $\text{Möb}_+(\mathbb{B}^n)$ extends canonically to a conformal automorphism of the sphere at infinity $\mathbb{S}^{n-1}_\infty := \overline{\mathbb{H}}^n_\mathbb{R} \setminus \mathbb{H}^n_\mathbb{R}$ and conversely, every conformal automorphism of $\mathbb{S}^{n-1}_\infty$ extends to an element in $\text{Möb}_+(\mathbb{B}^n)$.

1.3 Hyperbolic space

We now use Theorem 1.6 to construct a model for hyperbolic n-space $\mathbb{H}^n_\mathbb{R}$. This is Poincare's ball model for hyperbolic space.

We recall that a Riemannian metric g on a smooth manifold M means a choice of a positive definite quadratic form on each tangent space T_xM, varying smoothly over the points in M. Such a metric determines lengths of curves as usual, and so defines a metric on M in the usual way, by declaring the distance between two points to be the infimum of the lengths of curves connecting them.

Now consider the open unit ball $\mathbb{B}^n$, its tangent space $T_0\mathbb{B}^n$ at the origin, and fix the usual Riemannian metric on it, which is invariant under the action of $O(n)$. Given a point $x \in \mathbb{B}^n$, consider an element $\gamma \in \text{Möb}(\mathbb{B}^n)$ with $\gamma(0) = x$. Let $D\gamma_0$ denote the derivative at 0 of the automorphism $\gamma : \mathbb{B}^n \to \mathbb{B}^n$. This defines an isomorphism of vector spaces $D\gamma_0 : T_0\mathbb{B}^n \to T_x\mathbb{B}^n$ and allows us to define a Riemannian metric on $T_x\mathbb{B}^n$. In this way we get a Riemannian metric at each tangent space of $\mathbb{B}^n$.

We claim that the above construction of a metric on the open ball is well defined, *i.e.*, that the metric one gets on $T_x\mathbb{B}^n$ does not depend on the choice of the element $\gamma \in \text{Möb}(\mathbb{B}^n)$ taking 0 into x. In fact, if $\eta \in \text{Möb}(\mathbb{B}^n)$ is another element taking 0 into x, then $\eta^{-1} \circ \gamma$ leaves 0 invariant and is therefore an element in $O(n)$. Since the orthogonal group $O(n)$ preserves the metric at $T_0\mathbb{B}^n$, it follows that both maps, γ and η, induce the same metric on $T_x\mathbb{B}^n$. Hence this construction yields to a well-defined Riemannian metric on $\mathbb{B}^n$.

It is easy to see that this metric is complete and homogeneous with respect to points, directions and 2-planes, so it has constant (negative) sectional curvature.

Definition 1.8 The open unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ equipped with the above metric serves as a model for the *hyperbolic n-space* $\mathbb{H}^n_\mathbb{R}$. The group $\text{Möb}(\mathbb{B}^n)$ is its *group of isometries*, also denoted $\text{Iso}(\mathbb{H}^n_\mathbb{R})$, and its index two subgroup $\text{Möb}_+(\mathbb{B}^n)$ is the *group of orientation preserving isometries* of $\mathbb{H}^n_\mathbb{R}$, $\text{Iso}_+(\mathbb{H}^n_\mathbb{R})$.

In the sequel we denote the real hyperbolic space by $\mathbb{H}^n_{\mathbb{R}}$, to distinguish it from the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ (of real dimension $2n$) that we will consider later. Also, we denote by $\mathbb{S}^{n-1}_\infty$ the sphere at infinity, that is, the boundary of $\mathbb{H}^n_{\mathbb{R}}$ in $\mathbb{S}^n$. We set $\overline{\mathbb{H}}^n_{\mathbb{R}} := \mathbb{H}^n_{\mathbb{R}} \cup \mathbb{S}^{n-1}_\infty$.

Given that we have a metric in $\mathbb{H}^n_{\mathbb{R}}$, we can speak of length of curves, area, volume, and so on. We also have the concept of *geodesics*: curves that minimize (locally) the distance between points. These are the segments of curves in $\mathbb{H}^n_{\mathbb{R}}$ which are contained in circles that meet the boundary $\mathbb{S}^{n-1}_\infty$ orthogonally.

Notice that the constructions above show that every isometry of $\mathrm{Iso}(\mathbb{H}^n_{\mathbb{R}})$ extends canonically to a conformal automorphism of the sphere at infinity $\mathbb{S}^{n-1}_\infty$ and conversely, every conformal automorphism of $\mathbb{S}^{n-1}_\infty$ extends to an isometry of $\mathbb{H}^n_{\mathbb{R}}$.

Remark 1.9 (Models for real hyperbolic geometry) We remark that the model of hyperbolic space just described is particularly nice and useful in many ways. Yet, there are other models for hyperbolic space, each having its own interesting properties. There are several other classical models for real hyperbolic geometry: the projective ball model, the hyperboloid model, the upper-half sphere model and the Siegel domain model. The *upper-half sphere model* serves, among other things, to pass geometrically from the disc model to the upper-half space model and back. We refer to Thurston's book for descriptions of several other models for the hyperbolic n-space. These are briefly discussed below.

The hyperboloid model, also called *Lorentz or Minkowski model*, is very much related to the models we use in the sequel to study complex hyperbolic geometry (and so are the projective ball and the Siegel domain models that we describe below). For this we look at the upper hyperboloid $\mathcal{P}$ of the two-sheeted hyperboloid defined by the quadratic function

$$x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\,.$$

Its group of isometries is now $O(n,1)_o$, the subgroup of the Lorentz group $O(n,1)$ consisting of transformations that preserve $\mathcal{P}$. This model is generally credited to Poincaré too, though it seems that K. Weierstrass (and probably others) used it before. Its geodesics are the intersections of $\mathcal{P}$ with linear 2-planes in $\mathbb{R}^{n+1}$ passing through the origin; every linear space passing through the origin meets $\mathcal{P}$ in a totally geodesic subspace.

The projective ball model is also called the Klein, or Beltrami-Klein, model. For this we look at the disc $\mathcal{D}$ in $\mathbb{R}^{n+1}$ defined by

$$\mathcal{D} := \{(x_1, \cdots, x_{n+1}) \,\big|\, x_1^2 + \cdots + x_n^2 < 1 \text{ and } x_{n+1} = 1\,\}\,.$$

That is, we look at the points in $\mathbb{R}^{n+1}$ where the quadratic form $Q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$ is negative and $x_{n+1} = 1$. Notice that stereographic projection from the origin determines a bijection between $\mathcal{D}$ and $\mathcal{P}$.

Finally, *the Siegel domain (or paraboloid) model* for $\mathbb{H}^n_{\mathbb{R}}$ is obtained by looking at the points $(x_1, \cdots, x_n)$ in $\mathbb{R}^n$ that satisfy $2x_n > x_1^2 + \cdots + x_{n-1}^2$. This is bounded by a paraboloid, and it is equivalent to the upper half-space model. The *Cayley* transform provides an equivalence between this domain and the unit ball in $\mathbb{R}^n$.

1.4 Properly discontinuous actions

We assume for the moment that G is a group acting on a smooth manifold M by diffeomorphisms. Recall that the *stabiliser* of a point $x \in M$, also called the *isotropy*, is the subgroup $G_x \subset G$ defined by

$$G_x = \{g \in G \mid g(x) = x\}\,;$$

The *orbit* of x under the action of G is the set:

$$Gx = \{y \in M \mid y = g(x) \text{ for some } g \in G\}\,.$$

Definition 1.10 The action of G is *discontinuous at* $x \in M$ if there is a neighbourhood U of x such that the set

$$\{g \in G \mid gU \cap U \neq \varnothing\}$$

is finite. The set of points in M at which G acts discontinuously is called the *region of discontinuity*. This set is also called *the regular set* of the action. The action is *discontinuous on* M if it is discontinuous at every point in M.

One has the following well-known result.

Proposition 1.11 *If the G-action on M is discontinuous, then the G-orbits have no accumulation points in M. That is, if (g_m) is a sequence of distinct elements of G and $x \in M$, then the sequence $(g_m(x))$ has no limit points. Conversely, if G satisfies this condition, then G acts discontinuously on M.*

Definition 1.12 Let G act on the manifold M by diffeomorphisms. The action is said to be *properly discontinuous* if for each non empty compact set $K \subset M$ the set

$$\{g \in G \mid gK \cap K \neq \varnothing\}\,,$$

is finite.

It is clear that every properly discontinuous action is *a fortiori* discontinuous. The example below shows that the converse statement is false generally speaking.

The following example shows that discontinuous actions are not necessarily properly discontinuous:

Example 1.13 *Let G be the cyclic group induced by the transformation $g : \mathbb{C}^2 \to \mathbb{C}^2$ given by $g(z, w) = (\frac{1}{2}z, 2w)$. Clearly G acts discontinuously on $\mathbb{C}^2 \setminus \{0\}$, but if we let S be the set*

$$S = \{(z, w) \in \mathbb{C}^2 \big|\ |z| = |w| = 1\}\,,$$

then the set of cluster points of its orbit is $\{(z, w) \in \mathbb{C}^2 \mid z = 0\} \cup \{(z, w) \in \mathbb{C}^2 | w = 0\}$, the union of the two coordinate axis.

The following propositions provide equivalent ways of defining properly discontinuous actions:

Proposition 1.14 *The group G acts properly discontinuously on M if and only if for every pair of compact subsets K_1, K_2 of M, there are only a finite number of elements $g \in G$ such that $g(K_1) \cap K_2 \neq \emptyset$.*

Proposition 1.15 *Let G act properly discontinuously on M. Then the orbits of the action on compact sets have no accumulation points. That is, if (g_m) is a sequence of distinct elements of G and $K \subset M$ is a nonempty compact set, then the sequence $(g_m(K))$ has no limit points. Conversely, if G satisfies this condition, then G acts properly discontinuously on M.*

It is clear that the second proposition implies the previous one. We refer to [27] for the proof of 1.15.

Notice that if G acts discontinuously and freely on a manifold M, then the quotient map $\pi : M \to M/G$ is a covering map and the group of automorphisms of the covering is G itself, $Aut(M \to M/G) = G$. Yet, the example above shows that the quotient M/G may not be a Hausdorff space, even if the action is free and discontinuous. We notice that in this same example, the axis are the set of accumulation points of the orbits of compact sets in $\mathbb{C}^2 \setminus \{(0, 0)\}$. If we remove the axis, we get a a properly discontinuous action on their complement, and in that case the quotient is indeed Hausdorff. This is a general fact for properly discontinuous actions (see [27], [29]).

There is another related notion that will play a key-role in the sequel, so we introduce it now:

Definition 1.16 Let G be a group acting on a manifold X. *The equicontinuity region* of G, denoted $\mathrm{Eq}(G)$, is the set of points $x \in X$ for which there is an open neighbourhood U of x such that $G\,|_U$ is a normal family.

If we equip the manifold X with a Riemannian metric, the above definition is equivalent to saying that the family of transformations defined by G is equicontinuous at a point $x_0 \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(g(x_0), g(x)) < \epsilon$ for all $g \in G$

and all x such that $d(x_0, x) < \delta$. The family is equicontinuous if it is equicontinuous at each point of X.

Recall that a collection of transformations is a *normal family* if and only if every sequence of distinct elements has a subsequence which converges uniformly on compact sets.

1.5 Kleinian groups: The limit set

We now consider a subgroup $\Gamma \subset Iso(\mathbb{H}^n_{\mathbb{R}})$ and look at its action on the hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. We want to study how the orbits of points in $\mathbb{H}^n_{\mathbb{R}}$ (and in $\overline{\mathbb{H}}^n_{\mathbb{R}}$) behave under the action of Γ.

Definition 1.17 Let $\Gamma \subset \mathrm{Iso}(\mathbb{H}^{\mathrm{n}}_{\mathbb{R}})$ be a discrete subgroup. The *limit set* of Γ is the subset $\Lambda = \Lambda(\Gamma)$ of $\mathbb{S}^{n-1}_\infty$ of points which are accumulation points of orbits in $\mathbb{H}^n_{\mathbb{R}}$. That is,

$$\Lambda := \left\{ y \in \mathbb{S}^{n-1}_\infty \,|\, y = \lim\{g_m(x)\} \text{ for some } x \in \mathbb{H}^n_{\mathbb{R}} \text{ and } \{g_m\} \text{ a sequence in } \mathrm{Iso}(\mathbb{H}^{\mathrm{n}}_{\mathbb{R}}) \right\}.$$

By definition this is a closed invariant subset of $\mathbb{S}^{n-1}_\infty$ which is non-empty, unless Γ is finite. This is the set where the dynamics concentrates.

Definition 1.18 A discrete subgroup of $\mathrm{Iso}(\mathbb{H}^{\mathrm{n+1}}_{\mathbb{R}}) \cong \mathrm{Conf}(\mathbb{S}^{\mathrm{n}})$ is *Kleinian* if its limit set is not the whole sphere at infinity.

We remark that nowadays the term “Kleinian group” is being often used for an arbitrary discrete subgroup of hyperbolic motions, regardless of whether or not the region of discontinuity is empty.

Example 1.19 Consider an arbitrary family of pairwise disjoint closed 2-discs $D_1, ..., D_r$ in the 2-sphere with boundaries the circles $C_1, ..., C_r$. Let $\iota_1, ..., \iota_r$ be the inversions on these r circles, and let Γ be the subgroup of $\mathrm{Iso}(\mathbb{H}^3_{\mathbb{R}}) \cong \mathrm{Möb}(\mathbb{B}^3) \cong \mathrm{Conf}(\mathbb{S}^2)$ generated by these maps. Then G has nonempty region of discontinuity that contains the complement in $\mathbb{S}^2$ of the union $D_1 \cup ... \cup D_r$ (which is a fundamental domain for Γ). One can show that in this case the limit set is a Cantor set. This is an example of a *Schottky group*, and Schottky groups are all Kleinian groups.

Continuing with this example, move the discs $D_1, ..., D_r$ so that each of them touches tangentially exactly its two neighbors, and there is a common circle C orthogonal to all of them. Then C is the limit set of the corresponding group of inversions.

Now move the circles slightly $C_1, ..., C_r$, breaking the condition that they have a common orthogonal circle, keeping the condition that each disc touches with its two neighbors. Then one has (this is not at all obvious) that the limit set becomes a fractal curve of Hausdorff dimension between 1 and 2, and choosing appropriate deformations one can cover the whole range of Hausdorff dimension between 1 and 2. This is depicted in figure 1.19 below, and this is an example of a more general result by Rufus Bowen.

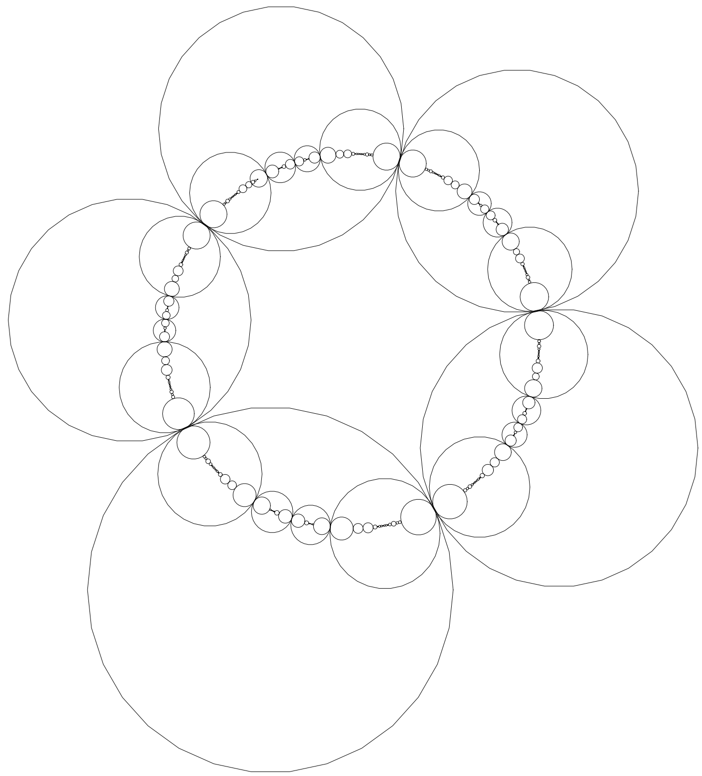

Figure 4: A "kissing Schottky" group with C as limit set.

These are all examples of Kleinian groups. So we see that whenever we have a Kleinian group, the sphere $\mathbb{S}^{n-1}_{\infty}$ splits in two sets, which are invariant under the group action: The limit set Λ, where the dynamics concentrates, and the region of discontinuity Ω where the dynamics is "mild" and plays an important role in geometry, as we will see later.

1.6 Some basic properties of the limit set.

We consider again a subgroup G of $\mathrm{Iso}(\mathbb{H}^{n+1}_{\mathbb{R}})$ and we think of it as acting on $\overline{\mathbb{H}}^{n+1}_{\mathbb{R}} := \mathbb{H}^{n+1}_{\mathbb{R}} \cup \mathbb{S}^n_{\infty}$. In this section we discuss some basic properties of the limit set of Kleinian groups.

Recall one has that the *region of discontinuity* of G is the set $\Omega = \Omega(G)$ of all points in $\overline{\mathbb{H}}^{n+1}_{\mathbb{R}}$ which have a neighbourhood that intersects only finitely many copies of its G-orbit.

The following property of the limit set follows easily from the fact that a Kleinian group acts by isometries on the hyperbolic space, so it cannot have accumulation points of the orbits within $\mathbb{H}^{n+1}_{\mathbb{R}}$. Yet, $\overline{\mathbb{H}}^{n+1}_{\mathbb{R}}$ is compact, so when considering the action on this space, there must me accumulation points.

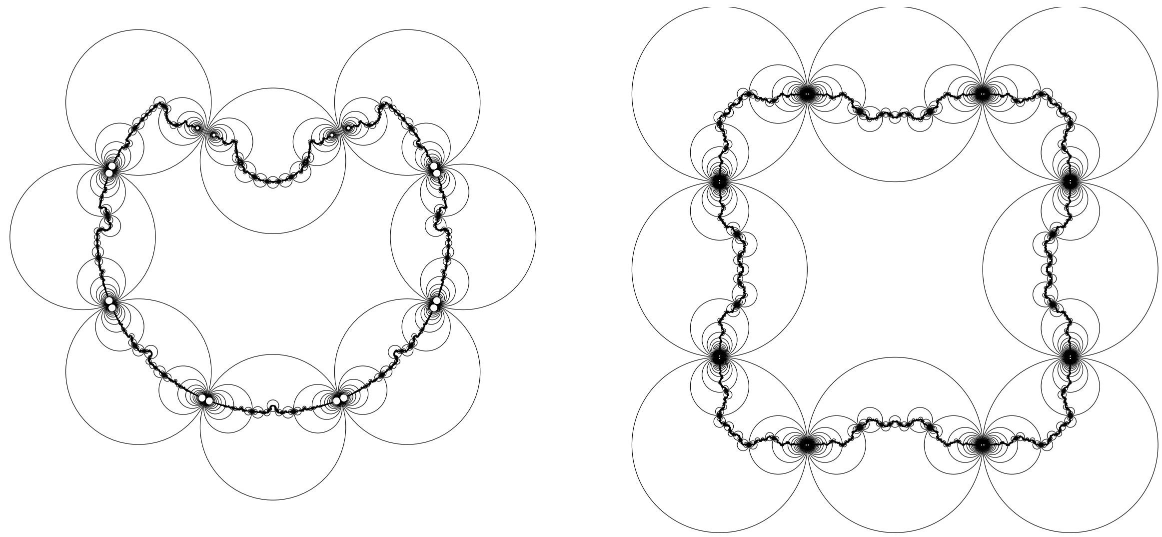

Figure 5: Deformation of a Fuchsian group: The limit set is a quasi-circle

Proposition 1.20 *Think of hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{R}}$ as being the unit $(n+1)$-ball equipped with the hyperbolic metric; its boundary is $\mathbb{S}^n$. Let (γ_m) be a sequence of distinct elements of a discrete group $G \subset \mathrm{Iso}(\mathbb{H}^{\mathrm{n}+1}_{\mathbb{R}})$. Then the set of accumulation points of the orbits $(\gamma_m)(x)$ is contained in $\mathbb{S}^n$.*

PROOF: Let us assume that the lemma is false and there is a subsequence of (γ_m), still denoted (γ_m), and $y \in \mathbb{H}^{n+1}_{\mathbb{R}}$ such that $\gamma_m(\infty) \xrightarrow[n\to\infty]{} y$.

We can equip $\mathbb{H}^{n+1}_{-} = \{x \in \mathbb{R}^{n+1} \mid |x| > 1\} \cup \{\infty\}$ with a metric of constant curvature -1 for which the group of preserving orientation isometries is given by $M\ddot{o}b(\mathbb{H}^{n+1}_{\mathbb{R}})$; denote such metric by d. Now let $z \in \mathbb{H}^{n+1}_{-}$, then

$$d(\gamma_m(z), y) \leq d(\gamma_m(z), \gamma_m(\infty)) + d(y, \gamma_m(\infty)) = d(z, x) + d(y, \gamma_m(x))\,.$$

Thus the set $\{\gamma_m(z) : m \in \mathbb{N}\}$ is relatively compact. Since $(\gamma_m) \subset \mathrm{Iso}(\mathbb{H}^{\mathrm{n}+1}_{-})$ the Arzelà-Ascoli theorem yields that there is a subsequence of (γ_m), still denoted by (γ_m), and $\gamma : \mathbb{H}^{n+1}_{-} \to \mathbb{H}^{n+1}_{-}$ such that $\gamma_m \xrightarrow[n\to\infty]{} \gamma$ in the compact-open topology. Clearly γ is an isometry and therefore G is nondiscrete. Which is a contradiction. $\square$

The following *convergence property* of Kleinian groups is fundamental, as from it spring several of the most basic properties of this theory. The same property holds in complex and quaternionic hyperbolic geometry and that is why the concept of limit set on those settings shares most of the properties one has in real hyperbolic geometry. This convergence property fails in general for arbitrary discrete subgroups of $\mathrm{PSL}(\mathrm{n}, \mathbb{C})$, $n > 2$, making that theory much more "intriguing" in some sense.

Lemma 1.21 (Convergence Property) *Let (γ_m) be a sequence of distinct elements of a discrete group $G \subset M\ddot{o}b(\mathbb{S}^n)$. Then either it contains a convergent subsequence, or it converges to a constant map away from a point in $\mathbb{S}^n$. That is, there exist a subsequence, still denoted (γ_m), and points $x, y \in \mathbb{S}^n$ such that:*

i. γ_m *converges uniformly to the constant function* y *on compact sets of* $\mathbb{S}^n - \{x\}$.

ii. γ_m^{-1} *converges uniformly to the constant function* x *on compact sets of* $\mathbb{S}^n - \{y\}$.

The proof of the convergence property is based on the existence and basic properties of isometric spheres (see for instance [15] for more details, or [25, 26] for a thorough account on the subject).

Theorem 1.22 *Let G be a subgroup of* $\mathrm{Iso}(\mathbb{H}^{n+1}_{\mathbb{R}})$. *The following three conditions are equivalent:*

i. The subgroup $G \subset \mathrm{Iso}(\mathbb{H}^{n+1}_{\mathbb{R}})$ *is discrete.*

ii. The region of discontinuity of G in $\mathbb{H}^{n+1}_{\mathbb{R}}$ is all of $\mathbb{H}^{n+1}_{\mathbb{R}}$.

iii. The region of discontinuity of G in $\mathbb{H}^{n+1}_{\mathbb{R}}$ is nonempty.

PROOF: It is clear that (ii) $\implies$ (iii) $\implies$ (i). Let us prove that (i) implies (ii). Let K be a compact set and assume that $K(G) = \{\gamma \in G : \gamma K \cap K \neq \emptyset\}$ is countable. Then by Lemma 1.21 there is a sequence $(\gamma_m) \subset K(G)$ and points $x, y \in \partial\mathbb{H}^{n+1}_{\mathbb{R}}$ such that γ_m converges uniformly to y on compact sets of $\overline{\mathbb{H}}^{n+1}_{\mathbb{R}} - \{x\}$. Let U be a neighbourhood of y disjoint from K. Then there is a natural number n_o such that $\gamma_m(K) \subset U$ for $m > n_o$. In particular we deduce $\gamma_m(K) \cap K = \emptyset$ for all $m > n_0$, which is a contradiction, and the result follows. □

Notice that by continuity, it is clear that if the region of discontinuity of G in $\mathbb{S}^n$ is nonempty, then the region of discontinuity of G in $\mathbb{H}^{n+1}_{\mathbb{R}}$ is nonempty and therefore G is discrete.

One has:

Theorem 1.23 *Let G be as above. Then the limit set is contained in the sphere at infinity $\mathbb{S}^n_\infty = \partial\mathbb{H}^{n+1}_{\mathbb{R}}$ and it is independent of the choice of orbit.*

PROOF: Let $x, y \in \mathbb{H}^{n+1}_{\mathbb{R}}$ and p a cluster point of Gy. Then there exists a sequence $(g_m) \subset G$ such that $g_m(y)$ converges to p. By Lemma 1.21 it follows that q also is a cluster point of $(g_m(x))$, which ends the proof. □

Theorem 1.24 *Let G be a discrete subgroup of* $\mathrm{Iso}(\mathbb{H}^{n+1}_{\mathbb{R}})$. *The limit set of G is the complement of the region of discontinuity in $\mathbb{S}^n_\infty$.*

PROOF: First we show that Λ lies in the complement of the region of discontinuity. Let $y \in \Lambda(G)$, then there is a point $p \in \mathbb{H}_{\mathbb{R}}^{n+1}$ and a sequence (γ_m) such that $\gamma_m(p) \to x$. From Lemma 1.21 we conclude that there exists $x \in \partial\mathbb{H}_{\mathbb{R}}^{n+1}$ such that we can assume that γ_m converges uniformly to the constant y on compact sets of $\overline{\mathbb{H}}_{\mathbb{R}}^{n+1} - \{x\}$. Let U be any neighbourhood of y. Then there is a natural number m_0 for which $\gamma_m(y) \in U$ for $m \geq m_0$.

Now let $q \in \partial\mathbb{H}_{\mathbb{R}}^{n+1}$ be a point in the discontinuity region, and assume that $q \in \Lambda(G)$. By the previous argument we deduce that q does not belong to the discontinuity region, which is a contradiction. Hence the discontinuity region is contained in the complement of $\Lambda(G)$. In others words, the complement of the discontinuity region is contained in $\Lambda(G)$. $\square$

It is clear from its definition that the limit set $\Lambda(G)$ is a closed G-invariant set, and it is empty if and only if G is finite (since every sequence in a compact set contains convergent subsequences).

Definition 1.25 Let G be a group acting on a manifold X. *The equicontinuity region* of G, denoted $\mathrm{Eq}(G)$, is the set of points $z \in X$ for which there is an open neighbourhood U of z such that $G\mid_U$ is a normal family.

Recall that a collection of transformations is a *normal family* if and only if every sequence of distinct elements has a subsequence which converges uniformly on compact sets.

One has:

Theorem 1.26 *Let G be a discrete subgroup of $\mathrm{Iso}(\mathbb{H}_{\mathbb{R}}^{n+1})$. Then the equicontinuity region of G coincides with the discontinuity region $\mathbb{S}^n \setminus \Lambda(G)$.*

PROOF: Observe that by Lemma 1.21, it is enough to show that $\mathrm{Eq}(G) \subset \Omega(G)$. Let $x \in \mathrm{Eq}(G)$ and assume that $x \in \Lambda(G)$, thus by Lemma 1.21 there is a sequence (γ_m) and a point y such that γ_m converges uniformly to the constant function y on compact sets of $\overline{\mathbb{H}}_{\mathbb{R}}^{n+1} - \{x\}$. Since $x \in \mathrm{Eq}(G)$, it follows that γ_m converges uniformly to the constant function y on $\overline{\mathbb{H}}_{\mathbb{R}}^{n+1}$. Let $q \in \mathbb{H}_{\mathbb{R}}^{n+1}$ and U be a neighbourhood of y such that $U \cap \mathbb{H}_{\mathbb{R}}^{n+1} \subset \mathbb{H}_{\mathbb{R}}^{n+1} - \{q\}$. The uniform convergence implies that there is a natural number n_0 such that $\gamma_m(\overline{\mathbb{H}}_{\mathbb{R}}^{n+1}) \subset U \cap \mathbb{H}_{\mathbb{R}}^{n+1} \subset \mathbb{H}_{\mathbb{R}}^{n+1} - \{q\}$ for each $m > m_0$. This is a contradiction since each γ_m is a homeomorphism. $\square$

We have:

Theorem 1.27 *Let G be discrete group such that its limit set has more than two points, then it has infinitely many points.*

PROOF: Assume that $\Lambda(G)$ is finite with at least 3 points. Thus $\widetilde{G} = \bigcap_{x\in\Lambda(G)} Isot(x,G)$ is a normal subgroup of G with finite index. Moreover, since each element in $\mathrm{Iso}(\mathbb{H}_{\mathbb{R}}^{n+1})$ has at most 2 fixed points in $\partial\mathbb{H}_{\mathbb{C}}^{n+1}$ we conclude that $\tilde{G}$ is trivial and therefore G is finite, which is a contradiction. □

Definition 1.28 The group G is *elementary* if its limit set has at most two points.

Theorem 1.29 *If G is not an elementary group, then its action on the limit set is minimal. That is, the closure of every orbit in $\Lambda(G)$ is all of $\Lambda(G)$.*

PROOF: Let $x, y \in \Lambda(G)$, then there is a sequence $(g_m) \subset G$ and a point $p \in \mathbb{H}_{\mathbb{R}}^{n+1}$ such that $g_m(p)$ converges to y. By Lemma 1.21 there is a point $q \in \partial\mathbb{H}_{\mathbb{R}}^{n+1}$, such that we can assume that g_m converges uniformly to y on compact sets of $\overline{\mathbb{H}}_{\mathbb{R}}^{n+1}$. Now, it is well known (see [4]) that there is a transformation $g \in G$ such that $g(x) \neq x$. thus we can assume that $x \neq q$ and therefore we conclude that $g_m(x)$ converges to y.

□

Corollary 1.30 *If G is a nonelementary Kleinian group, then $\Lambda(G)$ is a nowhere dense perfect set.*

In other words, if G is nonelementary then $\Lambda(G)$ has empty interior and every orbit in the limit set is dense in $\Lambda(G)$.

Remark 1.31 It is noticed in [35] that if the limit set of a nonelementary conformal group acting on $\mathbb{S}^n$ is a compact smooth k-manifold N, for some $0 < k \leq n$, then N is a round sphere $\mathbb{S}^k$. The proof, by Livio Flaminio, is a direct consequence, via stereographic projection of $\mathbb{S}^n$ into the tangent plane of $\mathbb{S}^n$ at a hyperbolic fixed point of the group, of the following fact: if M is a closed k-submanifold of $\mathbb{R}^n$ which is invariant under a homothetic transformation, then M is a k-dimensional subspace of $\mathbb{R}^n$.

2 COMPLEX KLEINIAN GROUPS

In the previous section we studied classical Kleinian groups; these are discrete subgroups of isometries of real hyperbolic spaces $\mathbb{H}^n_{\mathbb{R}}$. We remark that when $n = 3$, the sphere at infinity is 2-dimensional and we can think of it as being the Riemann sphere $\mathbb{S}^2$, which is a complex 1-dimensional manifold, diffeomorphic to the projective line $\mathbb{CP}^1$. Moreover, in this case one has that every (orientation preserving) element in the conformal group $\mathrm{Conf}_+(\mathrm{S}^2)$ is actually a Möbius transformation:

$$z \mapsto \frac{az+b}{cz+d}\,,$$

where a, b, c, d are complex numbers such that $ad - bz = 1$. The set of all such maps forms a group, which is isomorphic to the group $\mathrm{PSL}(2, \mathbb{C})$ of projective automorphisms of $\mathbb{CP}^1$:

$$\mathrm{PSL}(2, \mathbb{C}) := \mathrm{SL}(2, \mathbb{C})/ \pm \mathrm{Id}\,,$$

where $\mathrm{SL}(2, \mathbb{C})$ is the group of 2×2 matrices with complex coefficients and determinant 1, and Id is the identity matrix. Hence, considering discrete subgroups of $\mathrm{Iso}_+(\mathbb{H}^3_{\mathbb{R}})$ is the same thing as considering discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$.

We now look at discrete subgroups of $\mathrm{PSL}(\mathrm{n}+1, \mathbb{C})$, the group of automorphisms of the complex projective space $\mathbb{CP}^n$, and study complex Kleinian groups. This means a discrete subgroup of $\mathrm{PSL}(\mathrm{n}+1, \mathbb{C})$ that acts on the complex projective space $\mathbb{CP}^n$ with a nonempty region of discontinuity.

2.1 Complex projective space

We recall that the complex projective space $\mathbb{CP}^n$ is defined as:

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/ \sim\,,$$

where "$\sim$" denotes the equivalence relation given by $x \sim y$ if and only if $x = \alpha y$ for some nonzero complex scalar α. In short, $\mathbb{CP}^n$ is *the space of complex lines through the origin* in $\mathbb{C}^{n+1}$.

Consider for instance $\mathbb{CP}^1$. Every point here represents a complex line through the origin in $\mathbb{C}^2$. Recall that a complex line ℓ through the origin is always determined by a unit vector in it, say v, together with all its complex multiples. In other words, a unit vector v in $\mathbb{C}^2$ determines the complex line

$$\ell = \{\lambda \cdot v \,|\, \lambda \in \mathbb{C}\}\,.$$

Notice that the unit vectors in $\mathbb{C}^2$ form the 3-sphere $\mathbb{S}^3$, just as the unit vectors in $\mathbb{C}$ form the circle

$$\mathbb{S}^1 = \{z \in \mathbb{C} \,|\, z = e^{i\theta}\,, \theta \in [0, 2\pi]\}\,.$$

Notice that the circle $\mathbb{S}^1$ acts on $\mathbb{C}^2$ in the obvious way: $e^{i\theta}\cdot(z_1,z_2)\mapsto(e^{i\theta}z_1,e^{i\theta}z_2))$. This action preserves distances in $\mathbb{C}^2$, so given a point $v\in\mathbb{S}^3\subset\mathbb{C}^2$, its orbit under this $\mathbb{S}^1$-action is the set $\{(e^{i\theta}\cdot v\}$, which is a circle in $\mathbb{S}^3$ contained in the complex line determined by v. That is *the intersection of* $\mathbb{S}^3$ *with every complex line through the origin in* $\mathbb{C}^2$ is a circle, and one has:

$$\mathbb{CP}^1\cong\mathbb{S}^3/\mathbb{S}^1\cong\mathbb{S}^2\,.$$

The projection $\mathbb{S}^3\to\mathbb{CP}^1\cong\mathbb{S}^2$ is known as *the Hopf fibration*.

More generally, $\mathbb{CP}^n$ is a compact, connected, complex n-dimensional manifold, diffeomorphic to the orbit space $\mathbb{S}^{2n+1}/U(1)$, where $U(1)\cong\mathbb{S}^1$ is acting coordinate-wise on the unit sphere in $\mathbb{C}^{n+1}$. In fact, we usually represent the points in $\mathbb{CP}^n$ by *homogeneous coordinates* $(z_1:z_2:\cdots:z_{n+1})$. This means that we are thinking of a point in $\mathbb{CP}^n$ as being the equivalence class of the point $(z_1,z_2:\cdots,z_{n+1})$ up to multiplication by non-zero complex numbers. Hence if, for instance, we look at points where the first coordinate z_1 is not zero, then the point $(z_1:z_2:\cdots:z_{n+1})$ is the same as $(1:\frac{z_2}{z_1}:\cdots:\frac{z_{n+1}}{z_1})$. Notice this is just a copy of $\mathbb{C}^n$. That is, every point in $\mathbb{CP}^n$ that can be represented by a point $(z_1:z_2:\cdots:z_{n+1})$ with $z_1\neq 0$, has a neighbourhood diffeomorphic to $\mathbb{C}^n$, consisting of all points with homogeneous coordinates $(1:w_2:\cdots:w_{n+1})$. Of course similar remarks apply for points where $z_2\neq 0$ and so on. This provides the classical way for constructing an atlas for $\mathbb{CP}^n$ with $(n+1)$ coordinate charts.

Notice one has a projection $\mathbb{S}^{2n+1}\to\mathbb{CP}^n$, a *Hopf fibration*, and the usual Riemannian metric on $\mathbb{S}^{2n+1}$ is invariant under the action of $U(1)$. Therefore this metric descends to a Riemannian metric on $\mathbb{CP}^n$, which is known as the *Fubini-Study* metric.

It is clear that every linear automorphism of $\mathbb{C}^{n+1}$ defines a holomorphic automorphism of $\mathbb{CP}^n$, and it is well-known that every automorphism of $\mathbb{CP}^n$ arises in this way. Thus one has that the group of projective automorphisms is:

$$\mathrm{PSL(n+1,\mathbb{C})}\;:=\;\mathrm{GL(n+1,\mathbb{C})/(\mathbb{C}^*)^{n+1}}\cong\mathrm{SL(n+1,\mathbb{C})/\mathbb{Z}_{n+1}}\,,$$

where $(\mathbb{C}^*)^{n+1}$ is being regarded as the subgroup of diagonal matrices with a single nonzero eigenvalue, and we consider the action of $\mathbb{Z}_{n+1}$ (viewed as the roots of the unity) on $\mathrm{SL(n+1,\mathbb{C})}$ given by the usual scalar multiplication. Then $\mathrm{PSL(n+1,\mathbb{C})}$ is a Lie group whose elements are called projective transformations.

There is a classical way of decomposing the projective space that paves the way for studying complex hyperbolic geometry. For this we think of $\mathbb{C}^{n+1}$ as being a union $N_-\cup N_0\cup N_+$, where each of these sets consists of the points $(z_0,\cdots,z_n)\in\mathbb{C}^{n+1}$ satisfying that $|z_0|^2$ is, respectively, larger, equal or smaller than $|z_1|^2+\cdots+|z_n|^2$. It is clear that each of these sets is a complex cone, that is, union of complex lines through the origin in $\mathbb{C}^{n+1}$, with (deleted) vertex at 0.

Obviously

$$S:=\{(z_0,\cdots,z_n)\in N_0\,|\,z_0=1\,\}\,,$$

is a sphere of dimension $(2n-1)$, and N_0 is the union of all complex lines in $\mathbb{C}^{n+1}$ joining the origin $0 \in \mathbb{C}^{n+1}$ with a point in S; each such line meets S in a single point. Hence the projectivisation $[S] = (N_0 \setminus \{0\})/\mathbb{C}^*$ of N_0 is a $(2n-1)$-sphere in $\mathbb{CP}^n$ that splits this space in two sets, which are the projectivisations of N_- and N_+. The set N_0 is often called *the cone of light*.

Similarly, notice that the projectivisation of N_- is an open $(2n)$-ball $\mathbb{B}$ in $\mathbb{CP}^n$, bounded by the sphere $[S]$. This ball serves as model for complex hyperbolic geometry, as we will see in the following section, where we describe its full group of holomorphic isometries, which is naturally a subgroup of projective transformations. This gives a natural source of discrete subgroups of $\mathrm{PSL(n+1,\mathbb{C})}$, those coming from complex hyperbolic geometry.

Notice that the above decomposition of the projective space is being done according to the positive, negative and null sets of a quadratic form of signature -1. Of course there are similar decompositions corresponding to quadratic forms of different signatures: We shall discuss this below, when we speak about complex Lorentzian groups.

2.2 Complex Kleinian groups

Our aim in these lectures is to study discrete groups G of $\mathrm{PSL(n+1,\mathbb{C})}$ which act on $\mathbb{CP}^n$ with non-empty region of discontinuity. Recall from the previous section that the action of G is *discontinuous at* $x \in \mathbb{CP}^n$ if there is a neighbourhood U of x such that the set

$$\{g \in G \,|\, gU \cap U \neq \varnothing\}$$

is finite. The set of points in $\mathbb{CP}^n$ at which G acts discontinuously is called the *region of discontinuity*.

Definition 2.1 A discrete subgroup Γ of $\mathrm{PSL(n+1,\mathbb{C})}$ is *complex Kleinian* if its region of discontinuity in $\mathbb{CP}^n$ is non-empty.

This is a concept introduced by Alberto Verjovsky and José Seade some years ago (see SV1,SV2,SV3), which puts together several important areas of current research, as we shall see (we refer to [15] for more on the topic). For $n = 1$, $\mathbb{CP}^1$ is the Riemann sphere, $\mathrm{PSL}(2,\mathbb{C})$ can be regarded as being the group of (orientation preserving) isometries of the hyperbolic space $\mathbb{H}^3_{\mathbb{R}}$ and we are in the situation envisaged previously, of classical Kleinian groups.

Notice that in this classical case, there is a particularly interesting class of Kleinian subgroups of $\mathrm{PSL}(2,\mathbb{C})$: Those which are conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. This latter group can be regarded as the group of Möbius transformations with real coefficients:

$$z \mapsto \frac{az+b}{cz+d} \quad , \quad ad-bc = 1 \ , \ a,b,c,d \in \mathbb{R}\, .$$

These are the Möbius transformations that preserve the upper half plane in $\mathbb{C}$. And if we identify the Riemann sphere with the extended plane $\mathbb{C} \cup \infty$, via stereographic projection, these are the conformal automorphisms of the sphere that preserve the Southern hemisphere, *i.e.*, they leave invariant a 2-ball in $\mathbb{S}^2$. Equivalently, these are subgroups of $\operatorname{Iso} \mathbb{H}^3_{\mathbb{R}}$ which actually are groups of isometries of the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$. These are called *Fuchsian groups*. In higher dimensions, this role is played by the so-called complex hyperbolic groups. These are, by definition, subgroups of $\mathrm{PSL}(\mathrm{n}+1, \mathbb{R})$ which act on $\mathbb{CP}^n$ leaving invariant a certain open ball of complex dimension n, which serves as model for complex hyperbolic geometry. In the subsection below we speak a few words about this interesting subject. And in the following subsections we discuss various other types of Complex Kleinian Groups.

2.3 The Kulkarni limit set

We start our discussion about the limit set with an example, that illustrates how intricate this concept can be and why we bother about considering several possible definitions of it, which all coincide in the setting envisaged in Section 1.

Consider the following example from [31].

Example 2.2 Let $\gamma \in \mathrm{PSL}(3, \mathbb{C})$ be the projectivisation of the linear map $\tilde{\gamma}$ given by:

$$\tilde{\gamma} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

where $\alpha_1\alpha_2\alpha_3 = 1$ and $|\alpha_1| < |\alpha_2| < |\alpha_3|$. We denote by Γ the cyclic subgroup of $\mathrm{PSL}(3, \mathbb{C})$ generated by γ; we may choose the α_i so that Γ is conjugate to a subgroup of $\mathrm{PU}(1, 2)$. Denote by $\{e_1, e_2, e_3\}$ the usual basis of $\mathbb{C}^3$. Each of these vectors represents a complex line in $\mathbb{C}^3$ and therefore determines a point in $\mathbb{CP}^2$, that for simplicity we denote also by $\{e_1, e_2, e_3\}$; these are fixed points of the action, since the corresponding lines are invariant. The conditions on the norm of the eigenvalues imply that the backwards orbit of “almost” every point in $\mathbb{CP}^2$ converges to e_1, while most of the forward orbits converge all to e_3. To be precise, notice that since $\{e_1, e_2, e_3\}$ are fixed points, the lines joining them, $\overleftrightarrow{e_i, e_j}$, are invariant lines and the set of accumulation points of all orbits consists of the points $\{e_1, e_2, e_3\}$. The first of these is an attractor, the second is a saddle point and the latter is a source.

It is not hard to show that:

i. Γ acts discontinuously on $\Omega_0 = \mathbb{CP}^2 - (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_3, e_2})$, and also on $\Omega_1 = \mathbb{CP}^2 - (\overleftrightarrow{e_1, e_2} \cup \{e_3\})$ and $\Omega_2 = \mathbb{CP}^2 - (\overleftrightarrow{e_3, e_2} \cup \{e_1\})$.

ii. Ω_1 and Ω_2 are the maximal open sets where Γ acts properly discontinuously; and Ω_1/Γ and Ω_2/Γ are compact complex manifolds. (In fact they are Hopf manifolds).

iii. Ω_0 is the largest open set where Γ forms a normal family.

It follows that even if the set of accumulation points of the orbits consists of the points $\{e_1, e_2, e_3\}$, in order to actually get a properly discontinuous action we must remove a larger set. Furthermore, in this example we see that there is not a largest region where the action is properly discontinuous, since neither Ω_1 nor Ω_2 is contained in the other.

So one has several candidates to be called as "limit set":

- The points $\{e_1, e_2, e_3\}$ where all orbits accumulate. But the action is not properly discontinuous on all of its complement. Yet, this definition is good if we make this group conjugate to one in $\mathrm{PU}(1,2)$ and we restrict the discussion to the "hyperbolic disc" $\mathbb{H}^2_{\mathbb{C}}$ contained in $\mathbb{CP}^2$. This corresponds to taking the Chen-Greenberg limit set of Γ, that we shall define below.

- The two lines $\overleftrightarrow{e_1, e_2}$, $\overleftrightarrow{e_3, e_2}$, which are attractive sets for the iterations of γ (in one case) or γ^{-1} (in the other case). This corresponds to Kulkarni's limit set of Γ, that we define below, and it has the nice property that the action on its complement is properly discontinuous and also, in this case, equicontinuous. And yet, the proposition above says that away from either one of these two lines the action of Γ is discontinuous. So this region is not "maximal".

- Then we may be tempted to taking as limit set the complement of the "maximal region of discontinuity", but there is no such region: there are two of them, the complements of each of the two invariant lines, so which one we choose?

- Similarly we may want to define the limit set as the complement of "the equicontinuity region". In this particular example, that definition may seem appropriate. The problem is that this would rule out important cases, as for instance the Hopf manifolds, which can not be written in the form U/G where G is a discrete subgroup of $\mathrm{PSL}(3, \mathbb{C})$ acting equicontinuously on an open set U of $\mathbb{CP}^2$. Moreover, there are examples where Γ is the fundamental group of certain compact complex surfaces (Inoue surfaces) and the action of Γ on $\mathbb{CP}^2$ has no points of equicontinuity.

Thus one has different definitions with nice properties in different settings. Yet, it is clear that when considering actions on $\mathbb{CP}^n$ the classical definition of the limit set is not satisfactory because the action on its complement is not necessarily properly discontinuous. There is instead a refinement introduced by R. Kulkarni in [29]. This definition of limit set applies in a very general setting of discrete group actions, and it has the important property of assuring that its complement is an open invariant set where the group acts properly discontinuously. This is as follows. Recall that given a family $\{A_\beta\}$ of subsets of X, where β runs over some infinite indexing set B, a point $x \in X$ is a *cluster (or accumulation) point* of $\{A_\beta\}$ if every neighbourhood of x intersects A_β for infinitely many $\beta \in B$.

Given a space X and a group G as above, let $L_0(G)$ be the closure of the set of points in X with infinite isotropy group. Let $L_1(G)$ be the closure of the set of cluster points of orbits of points in $X - L_0(G)$, *i.e.*, the cluster points of the family $\{\gamma(x)\}_{\gamma\in G}$, where x runs over $X - L_0(G)$.

Finally, let $L_2(G)$ be the closure of the set of cluster points of $\{\gamma(K)\}_{\gamma\in G}$, where K runs over all the compact subsets of $X - \{L_0(G) \cup L_1(G)\}$. We have:

Definition 2.3 Let X be as above and G a group of homeomorphisms of X.

i. The *Kulkarni limit set* of G in X is the set

$$\Lambda_{\mathrm{Kul}}(G) := L_0(G) \cup L_1(G) \cup L_2(G).$$

ii. The *Kulkarni region of discontinuity* of G is

$$\Omega_{\mathrm{Kul}}(G) \subset X := X - \Lambda_{\mathrm{Kul}}(G).$$

It is easy to see that the set $\Lambda_{\mathrm{Kul}}(G)$ is closed in X and it is G-invariant (it can be empty). The set $\Omega_{\mathrm{Kul}}(G)$ (which also can be empty) is open, G-invariant, and G acts properly discontinuously on it.

When G is a Möbius (or conformal) group, the classical definitions of the *limit set* and the *discontinuity set* coincide with the above definitions.

In the sequel we will be looking at this limit set in comparison with other possible limit sets, as for instance the complement of the region of equicontinuity.

Let us describe some specially interesting particular types of complex Kleinian groups that illustrate the richness of the subject.

2.4 Complex hyperbolic groups

Let us look at the subset $[N_-]$ of $\mathbb{CP}^n$ consisting of points with homogeneous coordinates satisfying:

$$|z_0|^2 < |z_1|^2 + \cdots |z_n|^2 \,. \tag{2.4}$$

As noticed above, this set is an open ball $\mathbb{B}$ of real dimension $2n$ and its boundary,

$$[N_0] := \{(z_0 : \cdots : z_n) \in \mathbb{CP}^n \,|\, |z_0|^2 = |z_1|^2 + \cdots |z_n|^2\} \,,$$

is a sphere of real dimension $2n-1$. This set $[N_-]$ is the usual starting point for complex hyperbolic geometry; for this one needs to introduce a metric, which is known as the Bergman metric. We shall do that in a way similar to the one we used for real hyperbolic space.

Let $\mathrm{U}(n+1)$ be the unitary group. By definition, its elements are the $(n+1)\times(n+1)$ matrices which satisfy

$$\langle Uz, Uw\rangle = \langle z, w\rangle \,,$$

for all complex vectors $z = (z_0, ..., z_n)$ and $w = (w_0, ..., w_n)$, where $\langle \cdot, \cdot \rangle$ is the usual hermitian product on $\mathbb{C}^{n+1}$: $\langle z, w \rangle = \sum_{i=0}^{n} z_i \cdot \bar{w}_i$. This is equivalent to saying that the columns of U form an orthonormal basis of $\mathbb{C}^{n+1}$ with respect to the hermitian product.

We now let $\mathrm{U}(1,n)$ be the subgroup of $U(n+1)$ of transformations that preserve the quadratic form

$$Q(z_0, \cdots, z_n) = |z_0|^2 - |z_1|^2 - \cdots - |z_n|^2 \,. \tag{2.5}$$

In other words, an element $\mathrm{U} \in \mathrm{U}(n+1)$ is in $\mathrm{U}(1,n)$ if and only if $Q(z) = Q(\mathrm{U}z)$ for all points in $\mathbb{C}^{n+1}$. Let $\mathrm{PU}(1,\mathrm{n})$ be its projectivization. Then the action of $\mathrm{PU}(1,\mathrm{n})$ on $\mathbb{CP}^n$ leaves invariant the set $[N_-]$. To see this, recall that a point in $\mathbb{CP}^n$ is in $[N_-]$ if and only if its homogeneous coordinates satisfy equation (2.5). If $(z_0 : \cdots : z_n)$ is in $[N_-]$ and γ is in $\mathrm{PU}(1,\mathrm{n})$, then the point $\gamma(z_0 : \cdots : z_n)$ is again in $[N_-]$. Therefore the group $\mathrm{PU}(1,\mathrm{n})$ *acts on* the ball $[N_-] \cong \mathbb{B}^{2n}$.

Recall that to construct the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ we considered the unit open ball $\mathbb{B}^n$ in $\mathbb{R}^{n+1}$, and we looked at the action of the Möbius group $\text{Möb}_+(\mathbb{B}^n)$ on this ball. This action was transitive with isotropy $O(n,\mathbb{R})$. So we can consider the usual metric at the space $T_0(\mathbb{B}^n)$, tangent to the ball at the origin, and spread it around using that the action is transitive; we get a well-defined metric on the ball using the fact that the isotropy $O(n,\mathbb{R})$ preserves the usual metric.

Let us now do the analogous construction for the ball $[N_-]$ using the action of $\mathrm{PU}(1,\mathrm{n})$: It is an exercise to show that this action is transitive, with isotropy $\mathrm{PU(n)}$. Let P be the center of this ball, $P := (0 : 0 : \cdots : 0 : 1)$. We equip the tangent space $T_P([N_-]) \cong \mathbb{C}^n$ with the usual hermitian metric, and spread this metric around $[N_-]$ using the action of $\mathrm{PU}(1,\mathrm{n})$. Since the isotropy $\mathrm{PU(n)}$ preserves the metric in $T_P([N_-])$ we get a well-defined metric on the ball $[N_-] \cong \mathbb{B}^{2n}$. This is the Bergman metric on the ball $[N_-]$, which thus becomes a model for the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, with $\mathrm{PU}(1,\mathrm{n})$ as its group of holomorphic isometries. Its boundary $[N_0]$ is the sphere at infinity $\mathbb{S}^{2n-1}_{\infty}$.

Since the action of $\mathrm{PU}(1,\mathrm{n})$ on $\mathbb{H}^n_{\mathbb{C}}$ is by isometries, then one has (by general results of groups of transformations) that every discrete subgroup of $\mathrm{PU}(1,\mathrm{n})$ acts discontinuously on $\mathbb{H}^n_{\mathbb{C}}$. Hence, regarded as a subgroup of $\mathrm{PU(n+1)}$, such a group acts on $\mathbb{CP}^n$ with non-empty region of discontinuity. In other words, we have:

Every complex hyperbolic discrete group is a complex Kleinian group,

a statement that generalises to higher dimensions the well-known fact that every Fuchsian subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is Kleinian when regarded as a subgroup of $\mathrm{PSL}(2,\mathbb{C})$.

Consider now a discrete subgroup G of $\mathrm{PU}(1,\mathrm{n})$. As before, we take as model for complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ the ball $\mathbb{B} \cong \mathbb{B}^{2n}$ in $\mathbb{CP}^n$ consisting of points with homogeneous coordinates satisfying

$$|z_1|^2 + \cdots + |z_n|^2 < |z_0|^2 \,,$$

whose boundary is a sphere $\partial\mathbb{H}^n_{\mathbb{C}} \cong \mathbb{S}^{2n-1}_{\infty}$, and we equip $\mathbb{B}$ with the Bergman metric ρ to get $\mathbb{H}^n_{\mathbb{C}}$.

The following notion was introduced in [18].

Definition 2.6 If G is a discrete subgroup of $\mathrm{PU}(1,\mathrm{n})$, then its *Chen-Greenberg limit set*, denoted $\Lambda_{\mathrm{CG}}(G)$, is the set of accumulation points of the G-orbit of any point in $\mathbb{H}^n_{\mathbb{C}}$.

As remarked earlier in the real case, the fact that the action on $\mathbb{H}^n_{\mathbb{C}}$ is by isometries and G is discrete implies that the orbit of every $x \in \mathbb{H}^n_{\mathbb{C}}$ must accumulate in $\partial\mathbb{H}^n_{\mathbb{C}}$. Hence the limit set $\Lambda_{\mathrm{CG}}(G)$ is contained in the sphere at infinity, likewise in the conformal case. Moreover, one also has the following result of Chern-Greenberg, which is a consequence of the convergence property in complex hyperbolic geometry and the fact that the action is by isometries of $\mathbb{H}^n_{\mathbb{C}}$.

Proposition 2.7 *If the set $\Lambda_{\mathrm{CG}}(G) \subset X$ has more than two points, then every orbit in $\Lambda_{\mathrm{CG}}(G)$ is dense in $\Lambda_{\mathrm{CG}}(G)$.*

It is clear from the definition that $\Lambda_{\mathrm{CG}}(G)$ is a closed invariant subset of $\mathbb{S}^{2n-1}_{\infty}$, and the result above says that this set is minimal. In particular $\Lambda_{\mathrm{CG}}(G)$ does not depend on the choice of the orbit of the point in $\mathbb{H}^n_{\mathbb{C}}$.

Thus, when considering subgroups of $\mathrm{PU}(1,\mathrm{n})$ acting on the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, this definition of limit set is a good definition. Yet, if we consider the action of G on the whole projective space, it is easy to show that the action is not properly discontinuous away from the ball $[N_-] \subset \mathbb{CP}^n$, which serves as model for $\mathbb{H}^n_{\mathbb{C}}$. So we need to introduce another notion of limit set.

In the example 2.2 one has that the sets $L_0(G)$ and $L_1(G)$ are equal, and they consist of the three points $\{e_1, e_2, e_3\}$, while $L_2(G)$ consists of the lines $\overleftrightarrow{e_1, e_2}$ and $\overleftrightarrow{e_2, e_3}$, passing through the saddle point.

That example also shows that although Kulkarni's limit set has the property of assuring that the action on its complement is discontinuous, this region is not always maximal. Yet, one can show that in the case of actions of complex hyperbolic groups on $\mathbb{CP}^2$, "generically" this region is the largest open set where the action is discontinuous, and it coincides with the region of equicontinuity, by [31].

A first obvious question is to determine the relation between these two notions of limit set in the case of complex hyperbolic groups, as well as the relation of these sets with the corresponding region of equicontinuity.

One has the following theorem due to J. P. Navarrete [31] in the two dimensional case, and to A. Cano and J. Seade [17] in higher dimensions. For this, let $\mathbb{S}^{2n-1}_{\infty} \subset \mathbb{CP}^n$ be the boundary of $\mathbb{H}^n_{\mathbb{C}}$, so it is the sphere at infinity. Notice that for each point $x \in \mathbb{S}^{2n-1}_{\infty}$ there is a unique complex projective subspace $\mathcal{H}_x$ of complex dimension $n-1$ which is tangent to $\mathbb{S}^{2n-1}_{\infty}$ at x. Given a discrete subgroup $G \subset \mathrm{PU}(1,\mathrm{n})$, let $\mathcal{H}_G$ be the union of all these projective subspaces for all points in the limit set $\Lambda_{\mathrm{CG}}(G) \subset \mathbb{S}^{2n-1}_{\infty}$. This set is clearly G-invariant, since $\Lambda_{\mathrm{CG}}(G)$ is invariant and the G action on $\mathbb{S}^{2n-1}_{\infty}$ is by holomorphic transformations.

The following theorem was proved by J. P. Navarrete [] for $n = 2$, then extended partially to higher dimensions by A. Cano and J. Seade in [17] and completed recently, in full generality, by A. Cano, B. Liu and M. López in [14].

Theorem 2.8 *Let $G \subset \mathrm{PU}(1, \mathrm{n})$ be a discrete subgroup and let $Eq(G)$ be its equicontinuity region in $\mathbb{P}^n_{\mathbb{C}}$. Then $\mathbb{P}^n_{\mathbb{C}} \setminus Eq(G)$ is the union of all complex projective hyperplanes tangent to $\partial\mathbb{H}^n_{\mathbb{C}}$ at points in $\Lambda(G)$, and G acts properly discontinuously on $Eq(G)$. Moreover, $Eq(G)$ coincides with the Kulkarni region of discontinuity $\Omega_{\mathrm{Kul}}(G)$.*

We do not know yet whether or not in higher dimensions the Kulkarni region of discontinuity of G coincides with the region of equicontinuity.

2.5 Complex Lorentzian and Schottky groups

In analogy with the previous subsection, we now consider a quadratic form in $\mathbb{C}^{n+1}$ of signature $(n-q, q+1)$, say:

$$Q(z_0, ..., z_n) = -|z_0| - \cdots - |z_q| + |z_{q+1}| + \cdots |z_n| \,.$$

As before, we consider the sets of positive, null and negative vectors for this quadratic form, and consider their projectivisations $[N_+]$, $[N_0]$, $[N_-]$.

The image of the null set $[N_-]$ in $\mathbb{CP}^n$ consists of points with homogeneous coordinates satisfying:

$$|z_0|^2 + ... + |z_q|^2 = |z_{q+1}|^2 + \cdots |z_n|^2 \,. \tag{2.9}$$

We now consider the group $\mathrm{U}(q+1, n-q)$ of elements in $U(n+1)$ that preserve the above quadratic form, and we let $\mathrm{PU(q+1, n-q)}$ be its projectivisation. Then the action of $\mathrm{PU(q+1, n-q)}$ on $\mathbb{CP}^n$ leaves invariant the sets $[N_+]$, $[N_0]$, $[N_-]$. Notice that the null set $[N_0]$, which is a smooth real hypersurface of $\mathbb{CP}^n$ that splits this space in two parts, is a closed and hence compact submanifold of $\mathbb{CP}^n$. Thence, if Γ is a discrete subgroup of $\mathrm{PU(q+1, n-q)}$ with infinite cardinality, then every Γ-orbit in $[N_0]$ has accumulation points, and we can define a set:

$$\Lambda_0(Q) = \{[z] \in [N_0] \subset \mathbb{CP}^n \,\big|\, [z] \,\text{is an accumulation point of some orbit in}\, [N_0]\} \,.$$

Notice that at each $[z] \in [N_0]$ the tangent space of $[N_0]$, which is real of dimension $2n-1$, contains a unique complex subspace $\mathcal{L}_z$ of complex dimension $n-1$. Both sets $\Lambda_0(Q)$ and $\Lambda_K(Q) = \bigcup_{z \in \Lambda_0(Q)} \mathcal{L}_z$ are closed invariant sets.

If $q = 0$, so that we are in the previously envisaged setting of complex hyperbolic groups, then Theorem 2.8 says that $\Lambda_K(Q)$ is the Kulkarni limit set of Γ in $\mathbb{CP}^n$. Is this still true in general? I do not know the answer.

Let us consider now a special case of Lorentzian groups: the Schottky groups.

Classical Schottky groups in $\mathrm{PSL}(2,\mathbb{C})$ play a key role in both complex geometry and holomorphic dynamics. On the one hand, Köbe's retrosection theorem says that every compact Riemann surface can be obtained as the quotient of an open set in the Riemann sphere $\mathbb{S}^2$ which is invariant under the action of a Schottky group. On the other hand, the limit sets of Schottky groups have rich and fascinating geometry and dynamics, which has inspired much of the current knowledge we have about fractal sets and 1-dimensional holomorphic dynamics.

These are subgroups of $\mathrm{PSL}(2,\mathbb{C})$ obtained by considering disjoint families of circles in $\mathbb{CP}^1 \cong \mathbb{S}^2$. These circles play the role of *mirrors* that split the sphere in two diffeomorphic halves which are interchanged by a conformal map, and these maps generate the Schottky group.

In this subsection we briefly discuss a generalisation to higher dimensions studied in [36] of the classical Schottky groups . The idea is to construct appropriate *mirrors* in the complex projective space that split it in two parts which are interchanged by a holomorphic authomorphism, and then use these to construct discrete subgroups. This is possible only in odd dimensions. We refer to [36] for details (see also [15]).

Consider the subspaces of $\mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ defined by $L_0 := \{(a,0) \in \mathbb{C}^{2n+2}\}$ and $M_0 := \{(0,b) \in \mathbb{C}^{2n+2}\}$. Let S be the involution of $\mathbb{C}^{2n+2}$ defined by $S(a,b) = (b,a)$, which clearly interchanges L_0 and M_0.

The following elementary lemma is a starting point.

Lemma 2.10 *Let* $\Phi\colon \mathbb{C}^{2n+2} \to \mathbb{R}$ *be given by* $\Phi(a,b) = |a|^2 - |b|^2$. *Then:*

i. The set $E_S := \Phi^{-1}(0)$ *is a real algebraic hypersurface in* $\mathbb{C}^{2n+2}$ *with an isolated singularity at the origin* 0. *It is embedded in* $\mathbb{C}^{2n+2}$ *as a (real) cone over* $\mathbb{S}^{2n+1} \times \mathbb{S}^{2n+1}$*, with vertex at* $0 \in \mathbb{C}^{2n+2}$.

ii. This set E_S *is invariant under multiplication by* $\lambda \in \mathbb{C}$*, so it is in fact a complex cone that separates* $\mathbb{C}^{2n+2} \setminus \{(0,0)\}$ *in two diffeomorphic connected components* U *and* V*, which contain respectively* $L_0 \setminus \{(0,0)\}$ *and* $M_0 \setminus \{(0,0)\}$*. These two components are interchanged by the involution* S*, for which* E_S *is an invariant set.*

iii. Every linear subspace K *of* $\mathbb{C}^{2n+2}$ *of dimension* $n+2$ *containing* L_0 *meets transversally* E_S *and* M_0*. Therefore a tubular neighbourhood* V *of* $M_0 \setminus \{(0,0)\}$ *in* $\mathbb{CP}^{2n+1}$ *is obtained, whose normal disc fibres are of the form* $K \cap V$*, with* K *as above.*

Since E_S is a cone, we have that $[E_S \setminus \{0\}]_{2n+1}$ is a codimension 1 real submanifold of $\mathbb{CP}^{2n+1}$, that we denote simply by E_S. As a consequence of the previous lemma we have:

Corollary 2.11 *i.* E_S *is an invariant set of the involution* $[S]_{2n+1}$.

ii. E_S *is an* $\mathbb{S}^{2n+1}$*-bundle over* $\mathbb{CP}^n$*; in fact* E_S *is the sphere bundle associated to the holomorphic bundle* $(n+1)\mathcal{O}_{\mathbb{CP}^n}$*, which is the normal bundle of* $\mathbb{CP}^n$ *in* $\mathbb{CP}^{2n+1}$.

iii. E_S *separates* $\mathbb{CP}^{2n+1}$ *in two connected components which are interchanged by* $[S]_{2n+1}$ *and each one is diffeomorphic to a tubular neighbourhood of the canonical* $\mathbb{CP}^n$ *in* $\mathbb{CP}^{2n+1}$.

Definition 2.12 We call E_S *the canonical mirror* and $[S]_{2n+1}$ *the canonical involution.*

It is an exercise to show that Lemma 2.10 holds in the following more general setting. Of course one has the equivalent of Corollary 2.11 too.

Lemma 2.13 *Let* λ *be a positive real number and consider the involution*

$$S_\lambda : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \,,$$

given by $S_\lambda(a,b) = (\lambda b, \lambda^{-1} a)$. *Then* S_λ *also interchanges* L_0 *and* M_0, *and the set*

$$E_\lambda = \{(a,b) : |a|^2 = \lambda^2 |b|^2\}$$

satisfies, with respect to S_λ, *the analogous properties (i)-(iii)of Lemma 2.10 above.*

We notice that as λ tends to ∞, the manifold E_λ gets thinner and approaches the L_0-axes.

Consider now two arbitrary disjoint projective subspaces L and M of dimension n in $\mathbb{CP}^{2n+1}$. As before, we denote by $[\ ]_{2n+1}$ the natural projection $\mathbb{C}^{2n+2} \to \mathbb{CP}^{2n+1}$. It is clear that $\mathbb{C}^{2n+2}$ splits as a direct sum $\mathbb{C}^{2n+2} = [L]^{-1}_{2n+1} \oplus [M]^{-1}_{2n+1}$ and there is a linear automorphism H of $\mathbb{C}^{2n+2}$ taking $[L]^{-1}_{2n+1}$ to L_0 and $[L]^{-1}_{2n+1}$ to M_0. For every $\lambda \in \mathbb{R}_+$ the automorphism $[H^{-1} \circ S_\lambda \circ H]_{2n+1}$ is an involution of $\mathbb{CP}^{2n+1}$ that interchanges L and M.

Definition 2.14 A *mirror* in $\mathbb{CP}^{2n+1}$ means the image of the canonical mirror E_S under an element of $\mathrm{PSL}(2\mathrm{n}+2, \mathbb{C})$.

Observe that a mirror is the boundary of a tubular neighbourhood of a $\mathbb{CP}^n$ in $\mathbb{CP}^{2n+1}$, so it is an $\mathbb{S}^{2n+1}$-bundle over $\mathbb{CP}^n$. For $n = 0$ mirrors are just circles in $\mathbb{CP}^1 \cong \mathbb{S}^2$; one has that for $n = 1$ (and only in that dimension) this bundle is trivial, so in $\mathbb{CP}^3$ mirrors are copies of $\mathbb{S}^3 \times \mathbb{S}^2$. One has:

Lemma 2.15 *Let* L *and* M *be as above. Given an arbitrary constant* λ, $0 < \lambda < 1$, *we can find an involution* T *interchanging* L *and* M, *with a mirror* E *such that if* U^* *is the open component of* $\mathbb{CP}^{2n+1} \setminus E$ *which contains* M *and* $x \in U^*$, *then* $d(T(x), L) < \lambda d(x, M)$, *where the distance* d *is induced by the Fubini-Study metric.*

We notice that the parameter λ in 2.15 gives control upon the degree of expansion and contraction of the generators of the group, so one can estimate bounds on the Hausdorff dimension of the limit set.

The previous discussion can be summarised in the following theorem:

Theorem 2.16 *Let $L_1, \ldots, L_r$ be disjoint projective subspaces of $\mathbb{CP}^{2n+1}$ of dimension n, $r > 1$. Then:*

i. There exist involutions $T_1, \ldots, T_r$ of $\mathbb{CP}^{2n+1}$ with mirrors E_{T_j} such that if N_j denotes the connected component of $\mathbb{P}^{2n+1} \setminus E_{T_j}$ that contains L_j then $\{N_j \cup E_{T_j}\}$ is a closed family of pair-wise disjoint sets.

ii. $\Gamma = \langle T_1, \ldots, T_r \rangle$ is a discrete group with nonempty region of discontinuity.

iii. Given a constant $C > 0$, we can choose the T_j's so that if $T := T_{j_1} \cdots T_{j_k}$ is a reduced word of length $k > 0$ (i.e., $j_1 \neq j_2 \neq \cdots \neq j_{k-1} \neq j_k$), then $T(N_i)$ is a tubular neighbourhood of the projective subspace $T(L_i)$ which becomes very thin as k increases: $d(x, T(L_i)) < C\lambda^k$ for all $x \in T(N_i)$.

A group as in Theorem 2.16 was called *Complex Schottky* in [36]. It follows from the previous theorem that the set $W = \mathbb{CP}^{2n+1} \setminus \cup_{i=1}^{r} Int(N_i)$, where $Int(N_i)$ is the interior of the tubular neighbourhoods N_i as in Theorem 2.16, is a compact fundamental domain for the action of Γ on the open set $\Omega_{\mathrm{SV}}(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(W)$; and the action on Ω_S is properly discontinuous.

Definition 2.17 For a Schottky subgroup $\Gamma \subset \mathrm{PSL}(2\mathrm{n}+1, \mathbb{C})$ as above we may define $\Omega_{\mathrm{SV}}(\Gamma)$ to be the *SV-region of discontinuity.* Then we have the SV-limit set $\Lambda_{\mathrm{SV}} := \mathbb{CP}^n \setminus \Omega_{\mathrm{SV}}(\Gamma)$.

Theorem 2.18 *Let Γ be a complex Schottky group in $\mathbb{CP}^{2n+1}$, generated by involutions $\{T_1, \ldots, T_r\}$, $n \geq 1$, $r > 2$, as in Theorem (2.16) above. Let $\Omega_{\mathrm{SV}}(\Gamma)$ be the region of discontinuity of Γ and let $\Lambda_{\mathrm{SV}}(\Gamma) = \mathbb{CP}^{2n+1} \setminus \Omega_{\mathrm{SV}}(\Gamma)$ be the above limit set. Then:*

i. $\Lambda_{\mathrm{SV}}(\Gamma)$ is a complex solenoid (lamination) homeomorphic to $\mathbb{CP}^n \times \mathcal{C}$, where $\mathcal{C}$ is a Cantor set. Γ acts minimally on the set of projective subspaces in $\Lambda_{\mathrm{SV}}(\Gamma)$ considered as a closed subset of the Grassmannian $G_{2n+1,n}$.

ii. Let $\check{\Gamma} \subset \Gamma$ be the index 2 subgroup consisting of the elements which are reduced words of even length. Then $\check{W} = W \cup T_1(W)$ is a fundamental domain for the action of $\check{\Gamma}$ on $\Omega_{\mathrm{SV}}(\Gamma)$.

iii. Each element $\gamma \in \check{\Gamma}$ leaves invariant two copies, P_1 and P_2, of $\mathbb{CP}^n$ in $\Lambda_{\mathrm{SV}}(\Gamma)$. For every $L \subset \Lambda_{\mathrm{SV}}(\Gamma)$, $\gamma^i(L)$ converges to P_1 (or to P_2) as $i \to \infty$ (or $i \to -\infty$).

We now observe that one has the following, which is a particular case of a more general theorem due to V. Alderete, A. Cano and C. Cabrera:

Theorem 2.19 *The above Schottky groups are, up to conjugation, subgroups of* $\mathrm{PU}(\mathrm{n}, \mathrm{n})$.

It follows that one has for these the limit sets discussed in the previous section (for subgroups of $\mathrm{PU}(\mathrm{p}, \mathrm{q})$ in general). And it is natural to ask, in the case of the above Schottky groups, what is the relation of all these limit sets (It is clear that $\Lambda_{\mathrm{SV}}(\Gamma) \subset \Lambda_{\mathrm{Kul}}(\Gamma)$).

2.6 Kleinian groups and twistor theory

There is a construction by Seade and Verjovsky of complex Kleinian groups via twistor theory that we now recall. Twistor theory is an important area of geometry and mathematical physics, developed by various authors, most notably by Roger Penrose, in the late 1970s. There are also important contributions by M. Atiyah, N. Hitchin and several other authors. The idea is that each even-dimensional, oriented Riemannian manifold M has its twistor space $\mathfrak{Z}(M)$, a manifold which is a fibre bundle over M, and which under certain differential geometric restrictions on M, has a canonical complex structure. Furthermore, Penrose's twistor program springs from the fact that there is a rich interplay between the conformal geometry of the manifold M and the complex geometry of its twistor space. What Seade and Verjovsky did was showing that this interplay between the conformal geometry of M and the complex geometry of its twistor space can be pushed forward to dynamics. As a consequence we obtain that every conformal Kleinian group, or rather, every group of isometries of a real hyperbolic space (with non-empty region of discontinuity in the sphere at infinity) can be realised as a complex Kleinian group, *i.e.*, as a discrete group of holomorphic transformations of some complex projective space, with non-empty region of discontinuity.

This theory is particularly nice when the manifold M is the 4-sphere $\mathbb{S}^4$ endowed with its usual metric, and that is what we shall focus on in this section. The corresponding twistor space turns out to be the complex projective space $\mathbb{CP}^3$. This particular case is also relevant for other interesting problems in differential geometry, studied independently by E. Calabi. Hence in this case the twistor fibration:

$$\pi : \mathbb{CP}^3 \longrightarrow \mathbb{S}^4 \,,$$

is also known as the Calabi-Penrose fibration.

To construct this fibration, recall first that the complex projective line $\mathbb{CP}^1$ is the space of lines through the origin in $\mathbb{C}^2$, and so it is diffeomorphic to the sphere $\mathbb{S}^2$. We claim that, similarly, the sphere $\mathbb{S}^4$ is diffeomorphic to the quaternionic projective line $\mathcal{H}\mathbb{P}^1$. Let us explain this.

Recall that the complex numbers can be regarded as being $\mathbb{R}^2$ with a richer structure, coming from the fact that we have added the symbol i, which corresponds to the point $(0, 1)$ in $\mathbb{R}^2$, with $i^2 = -1$. Similarly, we have the space of quaternions $\mathcal{H}$. As a set, this is $\mathbb{R}^4$, a four-dimensional vector space over the real numbers, equipped with a richer structure, obtained by quaternionic multiplication. To define this multiplication we consider the usual basis of $\mathbb{R}^4$ and let $i = (0, 1, 0, 0)$, $j = (0, 0, 1, 0)$ and $k = (0, 0, 0, 1)$; we identify the scalar 1 with the vector $(1, 0, 0, 0)$. Then we define a multiplication by setting:

$$i^2 = j^2 = k^2 = ijk = -1$$

From this we get the well-known relations $ij = k; jk = i; ki = j$, and also $ij = -ji$ and so on. We extend this multiplication to all elements in $\mathcal{H}$ in the obvious way using that $1, i, j, k$ form a basis it as a vector space.

Notice that every quaternion can be expressed as:

$$q = a_0 + a_1 i + a_2 j + a_3 k = (a_0 + a_1 i) + (a_2 + a_3 i) j = z_1 + z_2 j \,.$$

So we see that every quaternion can be regarded as a pair of complex numbers, just as each complex number can be regarded as a pair of real numbers.

We consider now the space $\mathbb{C}^4$ and we identify it with $\mathcal{H} \times \mathcal{H} = \mathcal{H}^2$. Notice that we can multiply vectors in $v \in \mathbb{C}^4$ by complex numbers (scalars) in the usual way:

$$\lambda \cdot (z_1, z_2, z_3, z_4) = (\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4) \,.$$

Doing so, each vector $v \in \mathbb{C}^4$ determines a unique complex line ℓ_v in $\mathbb{C}^4$ passing through the origin:

$$\ell_v := \{ z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid z = \lambda\, v \quad , \text{ for some } \lambda \in \mathbb{C} \} \,.$$

Similarly, given a vector $v \in \mathcal{H}^2 \cong \mathbb{C}^4$, we can multiply it by quaternions, but one must decide to use either right or left multiplication (now this does matter, since this multiplication is non-commutative). In either case one gets, for each vector v, a quaternionic line $\mathcal{L}_v$, which is a 4-plane:

$$\mathcal{L}_v := \{ q = (q_1, q_2) \in \mathcal{H}^2 \mid q = \lambda\, v \quad , \text{ for some } \lambda \in \mathcal{H} \} \,.$$

Notice that each quaternionic line is actually a copy of $\mathbb{C}^2$ embedded in $\mathbb{C}^4$, spanned by the complex lines ℓ_v and ℓ_{jv}. In fact $\mathcal{L}_v$ is filled by complex lines.

Just as $\mathbb{CP}^3$ is obtained from $\mathbb{C}^4 \setminus 0$ by identifying points in the same complex line, so too we can form the quaternionic projective space:

$$\mathcal{H}\mathbb{P}^1 := \frac{\mathcal{H}^2 \setminus 0}{\mathcal{H}^*} \;\cong\; \mathbb{S}^7 / \mathbb{S}^3 \,,$$

the space of left quaternionic lines in $\mathcal{H} \times \mathcal{H}$. In other words, two non-zero quaternions q_1, q_2 are identified if there is another quaternion q such that $q\, q_1 = q_2$.

We leave it as an exercise to show that, just as one has:

$$\mathbb{RP}^1 \cong \mathbb{S}^1 \qquad \text{and} \qquad \mathbb{CP}^1 \cong \mathbb{S}^2 \,,$$

so too one has:

$$\mathcal{H}\mathbb{P}^1 \cong \mathbb{S}^4 \,.$$

Therefore we see that if in $\mathbb{C}^4 \cong \mathcal{H}^2$:

i) We identify each complex line to a point, then we get $\mathbb{CP}^3$;

ii) And if we identify each quaternionic line to a point, we get $\mathcal{H}\mathbb{P}^1 \cong \mathbb{S}^4$.

Since every complex line is contained in a unique quaternionic line, we thus get a projection map:

$$\pi : \mathbb{CP}^3 \longrightarrow \mathcal{H}\mathbb{P}^1 \,,$$

which is easily seen to be a locally trivial fibration, *i.e.*, a fibre bundle. For each point $[q_1 : q_2] \in \mathcal{H}\mathbb{P}^1$ the fiber $\pi^{-1}([q_1 : q_2])$ consists of all the complex lines through the origin in $\mathbb{C}^4 \cong \mathcal{H}^2$ which are contained in the same quaternionic line, which is a copy of $\mathbb{C}^2$. Hence each fibre is diffeomorphic to $\mathbb{CP}^1 \cong \mathbb{S}^2$.

This is the Calabi-Penrose fibration, also known as the twistor fibration of the 4-sphere.

We now recall that one has a group isomorphism:

$$\mathrm{Conf}_+(\mathbb{S}^2) \cong \left\{ \frac{\mathrm{az}+\mathrm{b}}{\mathrm{cz}+\mathrm{d}} \,\middle|\, \mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d} \in \mathbb{C} \right\} \cong \mathrm{PSL}(2,\mathbb{C})\,.$$

The proof of these facts can be adapted to showing the analogous statements (see Ahlfors' works [2, 3]):

$$\mathrm{Conf}_+(\mathbb{S}^4) \cong \left\{ (\mathrm{az}+\mathrm{b})(\mathrm{cz}+\mathrm{d})^{-1} \,\middle|\, \mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d} \in \mathcal{H} \right\} \cong \mathrm{PSL}(2,\mathcal{H})\,,$$

where the latter is the projectivisation of the group of 2×2 invertible matrices with coefficients in $\mathcal{H}$ and determinant one. Notice that one such matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts linearly on $\mathcal{H}^2$, and so it also acts on $\mathbb{C}^4 \cong \mathcal{H}^2$, with quaternionic multiplication being regarded as a 2×2 complex matrix. Hence there is a natural embedding

$$\mathrm{Conf}_+(\mathbb{S}^4) \hookrightarrow \mathrm{PSL}(4,\mathbb{C})\,.$$

Therefore we get:

Proposition 2.20 *Every group of orientation preserving isometries of the real hyperbolic space $\mathbb{H}^5_{\mathbb{R}}$ has a canonical lifting to a group of holomorphic automorphisms of the complex projective space $\mathbb{CP}^3$.*

$$\begin{array}{ccc} \mathbb{CP}^3 & \xrightarrow{\widetilde{\Gamma}} & \mathbb{CP}^3 \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathbb{S}^4 & \xrightarrow{\Gamma} & \mathbb{S}^4 \end{array}$$

This result is well-known in full generality (not only for the 4-sphere) for people working in twistor theory; this is also proved in [35] in a different way, using twistor theory. Notice that given an element $\gamma \in \mathrm{Conf}_+(\mathbb{S}^4)$, its lifting to $\mathrm{PSL}(4,\mathbb{C})$ is an automorphism of $\mathbb{CP}^3$ that carries fibres of π into fibres of π, and these are copies of $\mathbb{S}^2$. The fibres of π are called *twistor lines*, and it turns out that the action of Γ on $\mathbb{CP}^3$ carries twistor lines into twistor lines isometrically. Using this one can prove (see [35]):

Theorem 2.21 *Let $\Gamma \subset \mathrm{Conf}_+(\mathbb{S}^4)$ be a Kleinian group and let $\Omega(\Gamma) \subset \mathbb{S}^4$ be its region of discontinuity in the sphere. Denote by $\widetilde{\Gamma}$ its lifting to $\mathrm{PSL}(4,\mathbb{C})$. Then:*

- *The Kulkarni region of discontinuity* $\Omega_{\mathrm{Kul}}(\widetilde{\Gamma})$ *is* $\pi^{-1}(\Omega(\Gamma))$.
- *The action of* $\widetilde{\Gamma}$ *on the limit set* $\Lambda_{\mathrm{Kul}}(\widetilde{\Gamma}) := \mathbb{CP}^3 \setminus \Omega_{\mathrm{Kul}}(\widetilde{\Gamma})$ *is minimal if and only if* Γ *is either Zariski dense in* $\mathrm{Conf}_+(\mathbb{S}^4)$ *or else it is conjugate in* $\mathrm{Conf}_+(\mathbb{S}^4)$ *to a Zariski dense subgroup of* $\mathrm{Conf}_+(\mathbb{S}^3)$.
- *The quotient* $\Omega_{\mathrm{Kul}}(\widetilde{\Gamma})/\widetilde{\Gamma}$ *is an orbifold with a complex projective structure, and it is a manifold whenever* Γ *is torsion-free.*

Similar statements hold in higher dimensions (see [35]; also [36]).

3 THE LIMIT SET IN DIMENSION TWO

Let us recall first some facts about the limit set for classical Kleinian groups that we discussed already. The following theorem summarizes several of the fundamental properties of the limit set.

Theorem 3.1 *Let G be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Iso}(\mathbb{H}^3_{\mathbb{R}})$ of infinite cardinality, acting on the closed ball $\mathbb{H}^3_{\mathbb{R}} \cup \partial\mathbb{H}^3_{\mathbb{R}}$. Let Λ be its limit set and Ω its region of discontinuity in the sphere $\mathbb{S}^2$ (regarded as the boundary of the hyperbolic 3-space). Then:*

i. The set Λ is contained in $\mathbb{S}^2$.

ii. Ω and Λ are complementary: $\Omega = \mathbb{CP}^1 \setminus \Lambda$.

iii. The set Λ is closed, non-empty and G-invariant.

iv. The action on the region of discontinuity Ω actually is properly discontinuous;

v. If Λ has finite cardinality, then it consists of at most two points and the group is said to be elementary.

Furthermore, one has:

Theorem 3.2 *Assume G is non-elementary, then:*

i. Λ is minimal (every orbit is dense) and perfect.

ii. Ω is the largest set where the action is properly discontinuous;

iii. Ω is also the equicontinuity set for the G-action.

There is one more, very important, characterization of the limit set that comes from dynamics. For this we recall that every element in $\mathrm{PSL}(2,\mathbb{C})$ has a lifting to $\mathrm{SL}(2,\mathbb{C})$. The corresponding Jordan form is either of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} ,$$

for some non-zero complex number λ. In the first case the element in $\mathrm{PSL}(2,\mathbb{C})$ is said to be parabolic; in the second case it is elliptic if the eigenvalues have norm 1, or loxodromic otherwise.

We can describe this classification also by thinking of $\mathrm{PSL}(2,\mathbb{C})$ as being the group of Möbis transformation: Every such map is conjugate to either a translation of to a map of the form $z \mapsto \lambda z$ for some non-zero complex number λ; in the first case we have a parabolic element, and in the second case we have either an elliptic or a loxodromic element, depending on the norm of λ.

Notice that parabolic elements have only one fixed points, while every loxodromic element has two fixed points, one of these is attracting and the other repulsing. If we now look at the corresponding cyclic groups, the corresponding limit set consists of the fixed points, so it consists of either one or two points.

Elliptic elements are rotations; these also have two fixed points. The corresponding limit set is either empty, if the rotation is by a rational angle so that the group has finite order, or it is the whole sphere $\mathbb{S}^2$ if we are rotating by an irrational angle.

Coming back to groups, one has:

Theorem 3.3 *Let G be a non-elementary discrete subgroup of* $\mathrm{PSL}(2,\mathbb{C})$. *Then its limit set is the closure of the set of fixed points of loxodromic elements.*

In the sequel we describe how these properties extend to complex dimension 2. First we give a classification that works in general.

3.1 Classification of the elements in $\mathrm{PSL}(\mathrm{n}+1,\mathbb{C})$

The above classification of the elements in $\mathrm{PSL}(2,\mathbb{C})$ extends to higher dimensions as follows. We refer to [19] for the case of elements in $\mathrm{PU}(1,\mathrm{n})$. The case $n=2$ is proved in [32] and the general caseis proved in the recent paper [12] (see [10] for a thorough discussion of this classification). There are also several other related papers by Parker, Gongopadhyay and others. Here we restrict to what we need in the sequel and we refer to the bibliography for more on the topic.

We now discuss the classification in [12] of the elements in $\mathrm{PSL}(\mathrm{n}+1,\mathbb{C})$. We start with the definition.

Definition 3.4 Consider an element $g \in \mathrm{PSL}(\mathrm{n}+1,\mathbb{C})$. Then g is:

i. *Elliptic* if it has a lifting to $\mathrm{SL}(\mathrm{n}+1,\mathbb{C})$ which is diagonalizable with all eigenvalues of norm 1.

ii. *Parabolic* if it has a lifting to $\mathrm{SL}(\mathrm{n}+1,\mathbb{C})$ which is non-diagonalizable with all eigenvalues of norm 1.

iii. *Loxodromic* if it has a lifting to $\mathrm{SL}(\mathrm{n}+1,\mathbb{C})$ with at least one eigenvalue having norm $\neq 1$.

The classification theorem for the elements in $\mathrm{PSL}(2,\mathbb{C})$ extends beautifully to higher dimensions. Notice that the only proper projective subspaces of $\mathbb{CP}^1$ are points.

Theorem 3.5 *Let g be an element in* $\mathrm{PSL}(\mathrm{n}+1,\mathbb{C})$. *Then g is:*

- *Elliptic if and only if it either has finite order or else every point in $\mathbb{CP}^n$ is an accumulation point of some g-orbit.*
- *Parabolic if and only its Kulkarni limit set consists of exactly one proper projective subspace of $\mathbb{CP}^n$.*
- *Loxodromic if and only if its Kulkarni limit set consists of exactly two proper projective subspaces of $\mathbb{CP}^n$(which can have different dimensions).*

Remark 3.6 [The equicontinuity region] It is worth saying that in complex dimensions 1 and 2, the equicontinuity region of parabolic and loxodromic elements coincides with the Kulkarni region of discontinuity. In higher dimensions there exist parabolic and loxodromic elements for which the Kulkarni region of discontinuity and the region of equicontinuity do not coincide. We refer to [10] for an explicit example, and to [12] for further discussion on this topic.

Let us focus now on loxodromic elements in $\mathrm{PSL}(3,\mathbb{C})$. From the previous theorem we know that the Kulkarni limit set must consist of two projective subspaces of dimensions 0 or 1. In fact there are two possibilities: Λ_{Kul} either has two projective lines or one line and one point (see [32, 15, 10, 12]). For instance, with two lines in the limit set we already have Example (2.2), the projectivisation of the linear map $\tilde{\gamma}$ given by:

$$\tilde{\gamma} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix},$$

where $\alpha_1\alpha_2\alpha_3 = 1$ and $|\alpha_1| < |\alpha_2| < |\alpha_3|$.

The limit set Λ_{Kul} is the union of the two lines $\overleftrightarrow{e_1, e_2}$, $\overleftrightarrow{e_3, e_2}$, which are attractive sets for the iterations of γ (in one case) or γ^{-1} (in the other case).

The following is an example where Λ_{Kul} is the union of a line and a point. Let G be the cyclic group generated by the projectivization of the map:

$$\tilde{\gamma} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}, \text{ with } |\alpha| \neq 1\,.$$

Then $L_0(G) = L_1(G) = L_2(G)$ is the union of the line $\overleftrightarrow{e_1, e_2}$ and the point e_3. Hence $\Lambda_{\mathrm{Kul}}(G)$ is now:

$$\Lambda_{\mathrm{Kul}}(G) = \overleftrightarrow{e_1, e_2} \cup \{e_3\}\,.$$

Notice that in this example the invariant line is attracting and the fixed point is repelling or viceversa, depending on the norm of α.

As an example of a parabolic element consider the projectivisation of the map:

$$\tilde{\gamma} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This has 1 as the only eigenvalue. Now we have $L_0 = L_1 = \{e_1\}$, $L_2 = \overleftrightarrow{e_1, e_2}$, and $\Lambda_{\mathrm{Kul}}(G)$ consists of a single line:

$$\Lambda_{\mathrm{Kul}}(G) = \overleftrightarrow{e_1, e_2}.$$

3.2 The limit set in dimension two

One has the following theorem proved by W. Barrera, A. Cano and J. P. Navarrete. We recall that a family of projective lines in $\mathbb{CP}^2$ are said to be *in general position* if no three of them meet.

Theorem 3.7 *Let* Γ *be an infinite discrete subgroup of* $\mathrm{PSL}(3, \mathbb{C})$ *and let* $\Lambda_{\mathrm{Kul}} \subset \mathbb{P}^2_{\mathbb{C}}$ *be its Kulkarni limit set. Then:*

i. The set Λ_{Kul} *always contains at least one projective line.*

ii. The number of lines in Λ_{Kul} *is either 1, 2, 3 or infinite.*

iii. The number of lines in Λ_{Kul} *lying in general position is either 1, 2, 3, 4 or infinite.*

iv. The number of isolated points in Λ_{Kul} *is at most 1, and if there is one such point, then the group is virtually cyclic (generated by a loxodromic element) and* Λ_{Kul} *consists of 1 line and 1 point.*

The first statement in this theorem follows easily from the classification theorem of cyclic groups. This, together with statements (ii) and (iii) are proved in [8]. Statement (iv) is proved in [13] One ingredient of the proof is, in all cases, showing that there exist groups as stated: This is done explicitly. Then one must prove that there are no more possibilities, *i.e.*, that if the limit set has "enough lines" then it has infinitely many of them. The proof uses the theory of pseudo-projective transformations introduced in [17]. A key ingredient for that classifications is the following refinement in [6] of the classical Montel-Cartan theorem (cf. [28]).

Theorem 3.8 (Barrera-Cano-Navarrete) *Let* $\mathcal{F} \subset \mathrm{PSL}(3, \mathbb{C})$ *and* $\Omega \subset \mathbb{P}^2_{\mathbb{C}}$ *be a domain. If* $\bigcup_{f \in \mathcal{F}} f(\Omega)$ *omits at least* 3 *lines in general position in* $\mathbb{P}^2_{\mathbb{C}}$*, then* $\mathcal{F}$ *is a normal family in* Ω*.*

Notice that cyclic groups already provide examples where the number of lines in Λ_{Kul} is either 1 or 2, and also an example where it has 1 line and 1 point. The classification of groups with exactly one line in their limit set is given in [11], and that of groups with 2 lines is done in [13].

Examples of groups with exactly 3 lines in general position in their limit set are easy to construct. For instance take the example (2.2) with two lines in the limit set, and now introduce a new generator that permutes the points e_1, e_2, e_3; then the new limit set consists of the three invariant lines determined by these points. And the complete classification of the groups in $\text{PSL}(3, \mathbb{C})$ with at most three lines in their limit set is given in [13].

The construction of examples with exactly 4 lines in general position is not that simple and we refer to [7], where the authors also give the complete classification of such groups.

Examples of groups with infinitely many lines in general position in their limit set, and acting on $\mathbb{CP}^2$ with a non-empty region of discontinuity, are easy to provide. Such famillies, with rich dynamics, aregiven in [13] (see also [9]). Other interesting families are given in [16] and these are easy to describe: Consider first a cofinite Fuchsian group Γ in $\text{PU}(1, 1)$ and embed it in the obvious way in $\text{PU}(2, 1)$, so it acts on $\mathbb{CP}^2$. It leaves invariant a complex projective line $\mathcal{L}$ and it has a fixed point e_3 away from $\mathcal{L}$. Then the Kulkarni limit set consists of the pencil of projective lines joining e_3 with a point in $\mathcal{L}$. Notice that all these lines meet at e_3 so that Λ_{Kul} only has two lines in general position.

Now consider the same Fuchsian group Γ but think of it as a subgroup of $\text{SO}(2, 1)$ and embed it in $\text{PU}(2, 1)$ in the obvious way. It now leaves invariant the real Lagrangian plane $\mathcal{P} \cong \mathbb{RP}^2$ in $\mathbb{CP}^2$ of points with real homogeneous coordinates. In this case one can check (see [16]) that the limit set Λ_{Kul} has infinitely many lines in general position.

We may now state the theorem from [13] which says that in complex dimension 2, just as in dimension 1, the concept of limit set is well-defined generically. In dimension 1 we must rule out the cases when the limit set has "few" points to have the theorem saying that the limit set is a minimal set which is the closure of the set of fixed points of loxodromic elements, and its complement Ω is the largest set where the action is properly discontinuous and it coincides with the equicontinuity set. The corresponding statement in dimension 2 is given below. First we have:

Definition 3.9 Let G be a discrete subgroup of $\text{PSL}(3, \mathbb{C})$. We say that its action on $\mathbb{CP}^2$ is strongly irreducible if there is no a points nor projective lines in $\mathbb{CP}^2$ with finite orbit.

It is an exercise to show that this condition implies that the group has infinitely many lines in general position in its Kulkarni limit set (see [13]).

Theorem 3.10 *Let G be a discrete subgroup of* $\mathrm{PSL}(3,\mathbb{C})$ *with at least 3 lines in general position in its limit set Λ_{Kul}. Then:*

i. The set Λ_{Kul} is the closure of the set of attractive and repelling invariant lines of loxodromic elements.

ii. The set $\Omega_{\mathrm{Kul}} := \Lambda_{\mathrm{Kul}}$ is the largest set in $\mathbb{CP}^2$ where the action is properly discontinuous. Moreover, Ω_{Kul} also is the equicontinuity set for the G-action on $\mathbb{CP}^2$.

iii. The action on Λ_{Kul} may or may not be minimal, yet, if the action is strongly irreducible, then the induced action on the space of lines in $\mathbb{CP}^2$, i.e., the action on the dual $\check{\mathbb{CP}}^2$, is minimal on the dual set of Λ_{Kul}.

The table below summarizes the previous discussion.

Table 1: Dictionary between complex dimensions 1 and 2

The limit set Λ contains 1, 2 or ∞-many points	The Kulkarni limit set Λ_{Kul} contains 1, 2, 3 or ∞-many lines, and it contains 1, 2, 3, 4 or ∞-many lines in general position
The group is elementary if Λ has finite cardinality	The group is elementary if Λ_{Kul} has finitely many lines in general position
If the group is non-elementary: Λ is the closure of the set of fixed points of loxodromic elements	If the group is non-elementary: Λ_{Kul} is a union of lines and it is the closure of the set of invariant repulsive lines of loxodromic elements
Its complement Ω is the largest set where the action is properly discontinuous	Its complement Ω_{Kul} is the largest set where the action is properly discontinuous
Ω also is the region of equicontinuity	Ω_{Kul} also is the region of equicontinuity
The action on Λ is minimal	If there are no projective subspaces with finite orbit, then the action on the space of lines in Λ_{Kul} is minimal

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