Lorentzian Geometry II

Todd A. Drumm (Howard Univeristy, USA)

ICTP

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Deformations

- Reprsentations of a group G.
  - ϕ is a *representation*, $\phi : G \to \mathrm{GL}(k)$
  - ψ is an *affine representation*, $\psi : G \to \mathrm{Isom}(\mathrm{E}^{n,m}) = \mathrm{SO}(n,m) \ltimes \mathbf{R}^{n,m}$
- Linear map
  - Projection: $\mathbb{L} : \mathrm{Aff}(\mathrm{E}^{n,m}) \to \mathrm{GL}(n,m)$
  - $\mathbb{L}(A, \mathrm{a}) = A$
  - Set $\Gamma = \phi(G)$
- Cocycle map (not a homomorphism)
  - $u : \Gamma \to \mathbf{R}^{n,m}$
  - $u(A) = \mathrm{a}$ where A and a are as a above.
- *Deformation* of Γ is a continuous $\Phi_t : \mathrm{I} \times G \to \mathrm{GL}(n,m)$ such that Φ_t are all representations of G and $\Phi_0(G) = \Gamma$.

Affine Deformations

- An *affine deformation* of Γ is a map $\varphi : \Gamma \to \mathrm{Isom}(\mathsf{E}^{n,m})$ such that $\mathbb{L} \circ \varphi = \mathbb{I}$
  - Defined by *cocycle*, $u : \Gamma \to \mathbf{R}^{n,m}$
    - $\varphi(A) = (A, u(A))$.
    - $u(AB) = u(A) + Au(B)$
    - If φ_1 and φ_2 are conjugate by a translation v then $\delta_{\mathsf{v}} = u_1(A) - u_2(B) = \mathsf{v} - A(\mathsf{v})$ is a *coboundary*
  - Cohomology class of deformations
    - $Z(\Gamma, \mathbf{R}^{n,m})$ is all affine deformations of Γ, that is the set of all cocycles.
    - $B(\Gamma)$ is the seet of all coboundaries.
    - $H^1(\Gamma, \mathbf{R}^{n,m})$ describe translationally conjugate affine deformations of Γ

Three dimensions with $\text{rk}(\Gamma) = 2$

- First: Linear part
  - Suppose $\Gamma = \langle A, B \rangle = \langle A, B, C | ABC = Id \rangle$
  - Γ is determined by $(\text{tr}(A), \text{tr}(B), \text{tr}(C))$.
    - Up to conjugation.
    - Two of three generators can be mutlplied by -1.
    - Nielsen moves, e.g. $(A, B, C) \mapsto (A, B^{-1}, BA^{-1})$.

- Affine deformation φ of a fixed $\Gamma = \langle A, B, C | ABC = Id \rangle$
  - $\mathcal{A} = \varphi(A), \mathcal{B} = \varphi(B), \mathcal{C} = \varphi(C)$
  - Affine deformation defined by $(\alpha(\mathcal{A}), \alpha(\mathcal{B}), \alpha(\mathcal{C}))$
    - Up to translational conjugacy.
    - $H^1(\Gamma, \mathbf{R}^{2,1})$ is 3-dimensional.

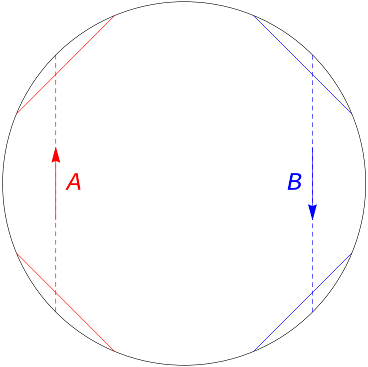
Three dimensions with $\mathrm{rk}(\Gamma) = 2$

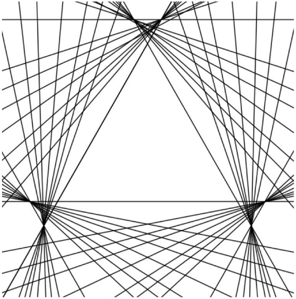
- Proper deformations of a group Γ, $\mathcal{P}(\Gamma)$
  - Projectiveness
    - $(A, k\mathrm{a}) \cdot (B, k\mathrm{b}) = (AB, k\mathrm{a} + A(k\mathrm{b})) = (AB, k(\mathrm{a} + A\mathrm{b}))$
    - $\alpha(A, k\mathrm{z}) = k\alpha(A, \mathrm{a})$
    - Affine deformation of Γ defined by cocycle u is proper if and only if affine deformation of Γ defined by ku is proper.

  - Opposite sign lemma implies that $\mathcal{P}(\Gamma)$ lie inside $(+, +, +)$ and the $(-, -, -)$.
    - Enough to consider intersection of $\mathcal{P}(\Gamma)$ with $x + y + z = 1$.
    - Plane defined by $\alpha(\mathcal{A}) = 0$ lies outside $\mathcal{P}(\Gamma)$.
    - Enough to consider 0-planse for (A, a) where $A \in \Gamma$ is *primitive*.

Proper Deformations of Three-Holed Spheres

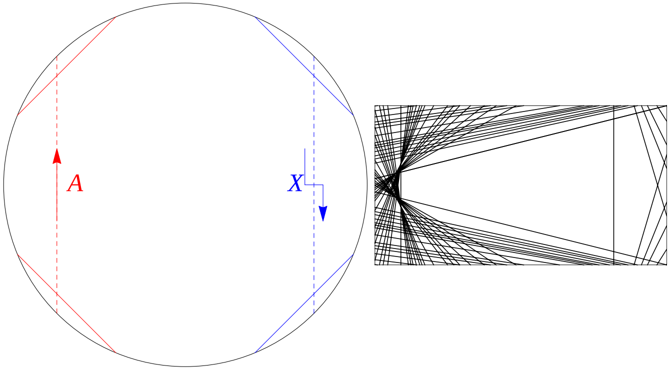
- H^2/Γ is a three holed sphere,
- $\Gamma = \langle A, B, C \rangle$, and A, B, C correspond to boundary closed geodesics.
- A deformation of Γ is proper if and only if $\alpha(\mathcal{A}), \alpha(\mathcal{B}), \alpha(\mathcal{C})$ are all the same sign.





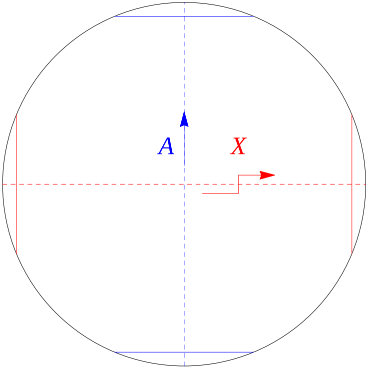
Proper Deformations of Two-holed Cross Surface

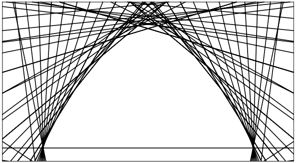
- H^2/Γ is a two-holed cross surface (unorianted).
- For proper deformation $\phi(\Gamma)$, $\mathsf{E}/\phi(\Gamma)$ is orientable.
- $\mathcal{P}(\Gamma)$ is four-sided.



Proper Deformations of One-holed Klein Bottle

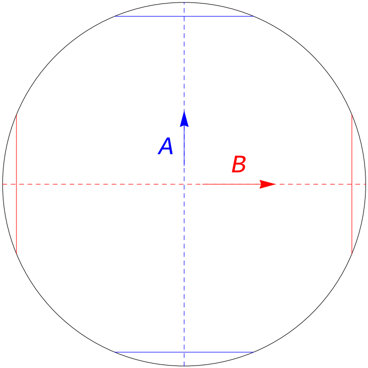
- H^2/Γ is a one-holed Klein bottle.
- $\mathcal{P}(\Gamma)$ has an infinite number of sides.

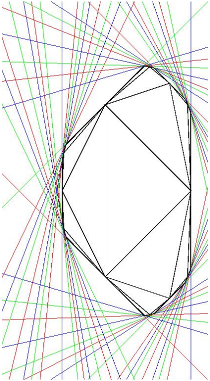




Proper Deformations of One-holed Torus

- H^2/Γ is a one-holed torus.
- $\mathcal{P}(\Gamma)$ has an infinite number of sides.





A Lie group and its Lie algebra

- $SL(2, \mathbf{R}) \cong SO(2,1)$
  - 2×2 real matrices with determinant 1.
  - Lie Group
- $\mathfrak{sl}(2, \mathbf{R})$
  - 2×2 real matrices with trace 0.
  - Lie algebra, tangent space to $SL(2, \mathbf{R})$ at $\mathbb{I}$.
  - Linear structure.
    - Killing form (multiple): $\mathbb{B}(\mathfrak{u}, \mathfrak{v}) = \frac{1}{2}\mathrm{tr}(\mathfrak{u}\mathfrak{v})$
    - $\mathfrak{e}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathfrak{e}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathfrak{e}_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
    - $\mathbf{R}^{2,1} \cong \mathfrak{sl}(2, \mathbf{R})$, where $(a, b, c) \mapsto a\mathfrak{e}_1 + b\mathfrak{e}_2 + c\mathfrak{e}_3$

Another view of the Margulis invariant

- $\mathrm{SL}(2,\mathbf{R}) \mapsto \mathfrak{sl}(2,\mathbf{R})$
  - $\mathfrak{gl}(2,\mathbf{R})$
    - The set of 2×2 real matrices.
    - $\mathrm{SL}(2,\mathbf{R}) \hookrightarrow \mathfrak{gl}(2,\mathbf{R})$, where $A \mapsto A$.
  - $\Pi : \mathfrak{gl}(2,\mathbf{R}) \to \mathfrak{sl}(2,\mathbf{R})$, where $\Pi(A) = A - \frac{\mathrm{tr}(A)}{2}\mathbb{I}$
    - Calculation for hyperbolic diagonal $A = \pm \begin{bmatrix} k & \\ & k^{-1} \end{bmatrix}$
    - $\Pi(A) = \pm\frac{1}{2}\begin{bmatrix} k - k^{-1} & \\ & k^{-1} - k \end{bmatrix}$
    - $\sqrt{\mathbb{B}(\Pi(A),\Pi(A))} = \sqrt{(\mathrm{tr}(A)^2 - 4)/4}$
    - $\frac{2\,\sigma\,\mathrm{tr}(A)}{\sqrt{\mathrm{tr}(A)^2-4}}\Pi(A) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} = A^0$, where $\sigma = \mathrm{tr}(\mathrm{sign}(A))$
- $\alpha(\mathcal{A}) = \frac{\mathrm{tr}(u(A)A)\cdot\sigma}{\sqrt{\mathrm{tr}(A)^2-4}}$
  - From: $\mathbb{B}(\mathcal{A}(O), A^0) = \mathrm{tr}\left(u(A)\frac{\sigma\,\mathrm{tr}(A)\left(A - \frac{\mathrm{tr}(A)}{2}\mathbb{I}\right)}{\sqrt{\mathrm{tr}(A)^2-4}}\right)$

Translation Length

- Let $\mu_t : G \to SL(2,\mathbf{R})$ be a smooth deformation where derivative at $A \in G$ is $u(A)$
  - $\tau_A := \left|\mathrm{tr}\left(A\left(\mathbb{I} + \left(tu(A) + O(t^2)\right)\right)\right)\right|$
  - $\frac{d\tau_A}{dt}(0) = \sigma \mathrm{tr}\left(Au(A)\right)$
  - Positve α corresponds to infinitesimal lengthening of a closed geodesic on the underlying surface.

- Results and Extensions
  - Goldman-Labourie-Margulis "extend α to a continuous function."
    - $\mathcal{C}(\Sigma)$ *geodesic currents* on a hyperbolic surface Σ.
    - Define $\Psi : \mathcal{C} \times H^1(\Gamma, \mathbf{R}^{2,1})$ which is continuous.
    - Result, if Ψ is positive then Γ acts properly on E.