Lorentzian Geometry III

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Euclidean Example

- The one-point compactification of $\mathbf{R}^n$:
  - Consider $\mathbf{R}^{n+1,1}$, that is the $(n+2)$–dimensional vector space with ...
    - the inner product $\langle \mathrm{v}, \mathrm{w} \rangle = v_1 w_1 + v_2 w_2 + \ldots + v_{n+1} w_{n+1} - v_{n+2} w_{n+2}$.
    - Or, $\langle \mathrm{v}, \mathrm{w} \rangle = \mathrm{v}^T \left(\mathbb{I}_{n+1} \oplus -\mathbb{I}_1 \right) \mathrm{w}$
  - The *null cone* $N^{n+1,1} = \left\{ \mathrm{v} \in \mathbf{R}^{n+1,1} \middle| \langle \mathrm{v}, \mathrm{v} \rangle = 0 \right\}$
  - *The conformal Riemannian sphere* $S^n = \mathbb{P}(N^{n+1,1})$, where $\mathbb{P}$ denotes projectivization.
    - Equivalence relation between nonzero vectors wehere $\mathrm{v} \sim k\mathrm{v}$.
    - Recall $(\mathrm{v}) = [v_1 : v_2 : \ldots : v_{n+1}]$

Euclidean Example

- Embedding Euclidean space $\mathbf{R}^n$ in S^n.
  - A quadratic form of $\mathbf{R}^{n,1}$: $\mathbb{I}_n \oplus \left(-\frac{1}{2}\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
  - In this form, $\langle \mathrm{v}, \mathrm{v} \rangle = v_1^2 + \ldots + v_n^2 - v_{n+1} v_{n+2}$
  - $\mathbf{R}^n \hookrightarrow N^{n+1,1}$ is $\mathrm{x} \mapsto \begin{bmatrix} \mathrm{x} \\ \mathrm{x} \cdot \mathrm{x} \\ 1 \end{bmatrix}$, where $\cdot$ denotes the Euclidean inner product in n-dimensions
  - The point missed is the class of $\begin{bmatrix} 0_n \\ 1 \\ 0 \end{bmatrix}$

Euclidean Example: Tranformations of the embedding

- Transformations
  - $\mathrm{O}(n+1,1)$ leaves invariant $N^{n+1,1}$
  - $\mathrm{PO}(n+1,1)$ is the set of conformal automorphisms of S^n.

- *Euclidean similarity transformations*
  - $T(\mathrm{x}) = rA\mathrm{x} + \mathrm{y}$, where $r \in \mathbf{R}_+$, $A \in \mathrm{O}(n)$ and $\mathrm{y} \in \mathbf{R}^n$
  - In $\mathrm{O}(n+1,1)$, $\begin{bmatrix} \mathbb{I}_n & 0 & \mathrm{y} \\ 2\mathrm{y}^T & 1 & \mathrm{y}\cdot\mathrm{y} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1/r \end{bmatrix}$
  - $\begin{bmatrix} A & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1/r \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{x}\cdot\mathrm{x} \\ 1 \end{bmatrix} = \begin{bmatrix} A\mathrm{x} \\ r\mathrm{x}\cdot\mathrm{x} \\ 1/r \end{bmatrix} \sim \begin{bmatrix} rA\mathrm{x} \\ r^2\mathrm{x}\cdot\mathrm{x} \\ 1 \end{bmatrix}$
  - $\begin{bmatrix} \mathbb{I}_n & 0 & \mathrm{y} \\ 2\mathrm{y}^T & 1 & \mathrm{y}\cdot\mathrm{y} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{x}\cdot\mathrm{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{x}+\mathrm{y} \\ (\mathrm{x}+\mathrm{y})\cdot(\mathrm{x}+\mathrm{y}) \\ 1 \end{bmatrix}$
    (where $2\mathrm{y}^T\mathrm{x} = 2\mathrm{y}\cdot\mathrm{x} = \mathrm{x}\cdot\mathrm{y} + \mathrm{y}\cdot\mathrm{x}$)

Euclidean Example: Tranformations of the embedding

- Inversion in the unit sphere: $\iota = \mathbb{I}_n \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
  - For $x \in \mathbf{R}^n$, $\iota : \begin{bmatrix} x \\ x \cdot x \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} x \\ 1 \\ x \cdot x \end{bmatrix} \sim \begin{bmatrix} x/(x \cdot x) \\ 1/(x \cdot x) \\ 1 \end{bmatrix}$.
  - In $\mathbf{R}^n$, $x \mapsto \frac{1}{x \cdot x} x$
  - The origin in homogenous coordinates: $\begin{bmatrix} 0_n \\ 0 \\ 1 \end{bmatrix}$
    - $\iota\left(\begin{bmatrix} 0_n \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0_n \\ 1 \\ 0 \end{bmatrix} = \text{“}\infty\text{''}$

The Lorentzian Compactification

- Consider $\mathbf{R}^{n+1,2}$
  - The convenient form of the inner product $\langle \mathrm{v}, \mathrm{w} \rangle = \mathrm{v}^T \left(\mathbb{I}_n \oplus -\mathbb{I}_l \oplus \left(-\frac{1}{2}\right) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \mathrm{w}$
    - $\langle \mathrm{v}, \mathrm{v} \rangle = v_1^2 + \ldots + v_n^2 - v_{n+1}^2 - v_{n+2} v_{n+3}$
  - The *null cone* $N^{n+1,2} = \left\{ \mathrm{v} \in \mathbf{R}^{n+1,1} \middle| \langle \mathrm{v}, \mathrm{v} \rangle = 0 \right\}$
- *The Einstein Universe* $\mathrm{Ein}^{n,1} = \mathbb{P}(N^{n+1,2})$.
- Embedding Lorentzian space $\mathbf{R}^{n+1}$ into $\mathrm{Ein}^{n,1}$
  - $\mathrm{x} \mapsto \begin{bmatrix} \mathrm{x} \\ \mathrm{x} \cdot \mathrm{x} \\ 1 \end{bmatrix}$ (here the "$\cdot$" is the Lorentzian inner product.)

Tranformations

- $\mathrm{PO}(n+1,2)$ is the set of conformal automorphisms of $\mathrm{Ein}^{n,1}$.
- *Lorentzian similarity transformations*
  - $T(\mathrm{x}) = rA\mathrm{x} + \mathrm{y}$, where $r \in \mathbf{R}_+$, $A \in \mathrm{O}(n,1)$, and $\mathrm{y} \in \mathbf{R}^{n,1}$
  - In $\mathrm{O}(n+1,2)$, $\begin{bmatrix} Q & 0 & \mathrm{y} \\ 2\mathrm{v}^T Q & 1 & \mathrm{y}\cdot\mathrm{y} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1/r \end{bmatrix}$, where $Q = \mathbb{I}_n \oplus -\mathbb{I}_1$ is the matrix representing the standard Lorentzian inner product.
- The *Lorentzian inversion in the unit sphere*

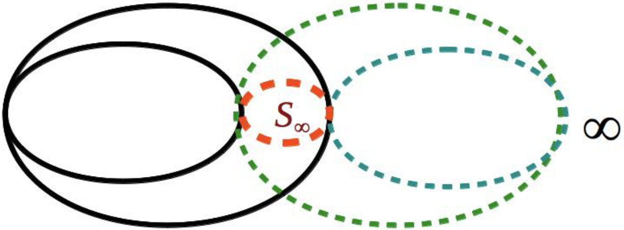
  $\iota = \mathbb{I}_{n+1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

  - For $\mathrm{x} \in \mathbf{R}^{n,1}$, $\iota\left(\begin{bmatrix} \mathrm{x} \\ \mathrm{x}\cdot\mathrm{x} \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \mathrm{x} \\ 1 \\ \mathrm{x}\cdot\mathrm{x} \end{bmatrix} \sim \begin{bmatrix} \mathrm{x}/(\mathrm{x}\cdot\mathrm{x}) \\ 1/(\mathrm{x}\cdot\mathrm{x}) \\ 1 \end{bmatrix}$.
  - In $\mathbf{R}^{n,1}$, $\mathrm{x} \mapsto \frac{1}{\mathrm{x}\cdot\mathrm{x}}\mathrm{x}$

Points at infinity

- *The Cone at infinity* C_∞
  - *The improper point* (or *destiny*):
    $$\infty = \iota\left(\begin{bmatrix} 0_{n+1} \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0_{n+1} \\ 1 \\ 0 \end{bmatrix}$$
  - Generic points: Given $\mathrm{x} \in N^{n,1}$, $\iota\left(\begin{bmatrix} \mathrm{x} \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \mathrm{x} \\ 1 \\ 0 \end{bmatrix}$.
- *The ideal sphere* S_∞: Given $\mathrm{x} \in N^{n,1}$, $\begin{bmatrix} \mathrm{x} \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} k\mathrm{x} \\ 0 \\ 0 \end{bmatrix}$, for $k \in \mathbf{R}$.
  - $\cong S^{n-1}$
  - Fixed pointwise by ι.

Points at infinity



Extending lines in $\mathbf{R}^{n,1}$ to $\mathrm{Ein}^{n,1}$

- Recall for $x \in \mathbf{R}^{2,1}$, $\iota(x) = \begin{bmatrix} x & x \cdot x & 1 \end{bmatrix}^T$
- Lines through the origin
  - Lightlike lines: $\iota\left(\begin{bmatrix} 0 \\ t \\ t \end{bmatrix}\right) = \begin{bmatrix} 0 \\ t \\ t \\ 0 \\ 1 \end{bmatrix} \xrightarrow[t\to\infty]{} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in S_\infty$
  - Timelike lines: $\iota\left(\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ t \\ -t^2 \\ 1 \end{bmatrix} \xrightarrow[t\to\infty]{} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \infty$
  - Spacelike lines: $\iota\left(\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} t \\ 0 \\ 0 \\ t^2 \\ 1 \end{bmatrix} \xrightarrow[t\to\infty]{} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \infty$

Extending generic lines in $\mathbf{R}^{n,1}$ to $\text{Ein}^{n,1}$

- Lightlike lines:

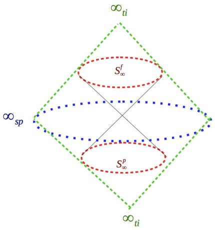
$$\iota\left(\begin{bmatrix} a \\ b+t \\ c+t \end{bmatrix}\right) = \begin{bmatrix} a \\ b+t \\ c+t \\ k+2(b-c)t \\ 1 \end{bmatrix} \xrightarrow[t\to\infty]{} \begin{bmatrix} 0 \\ \frac{1}{2(b-c)} \\ \frac{1}{2(b-c)} \\ 1 \\ 0 \end{bmatrix} \in C_\infty$$

where $k = a^2 + b^2 - c^2$

- Spacelike lines:

$$\iota\left(\begin{bmatrix} a+t \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+t \\ b \\ c \\ k+2at+t^2 \\ 1 \end{bmatrix} \xrightarrow[t\to\infty]{} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \infty$$

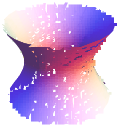
The double cover, $\widehat{\text{Ein}}$



- $\widehat{\text{Ein}}^{n,1} \cong S^n \times S^1$
- $\text{Ein}^{n,1} = \widehat{\text{Ein}}^{n,1} / \pm 1$ is
    - orientable if n is odd
    - nonorientable if n is even

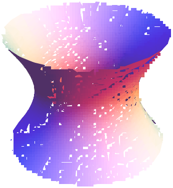
The big picture

- Constant curvature spaces
  - In $\mathbf{R}^{n,1}$
    - $\mathrm{H}^n = \{\mathrm{v} \in \mathbf{R}^{n,1} | \mathrm{v} \cdot \mathrm{v} = -1\} \cong \mathbb{P}\left(\{\mathrm{v} \in \mathbf{R}^{n,1} | \mathrm{v} \cdot \mathrm{v} < 0\}\right)$: curvature -1
    - ...bounded by $S^{n-1} = \mathbb{P}\left(\{\mathrm{v} \in \mathbf{R}^{n,1} | \mathrm{v} \cdot \mathrm{v} = 0\}\right)$
    - which bounds $\{\mathrm{v} \in \mathbf{R}^{n,1} | \mathrm{v} \cdot \mathrm{v} = 1\} \cong \mathbb{P}\left(\{\mathrm{v} \in \mathbf{R}^{n,1} | \mathrm{v} \cdot \mathrm{v} > 0\}\right)$: curvature $+1$
  - In $\mathbf{R}^{n+1,2}$
    - $\mathrm{AdS}^{n,1} = \{\mathrm{v} \in \mathbf{R}^{n+1,2} | \mathrm{v} \cdot \mathrm{v} = -1\} \cong \mathbb{P}\left(\{\mathrm{v} \in \mathbf{R}^{n,1} | \mathrm{v} \cdot \mathrm{v} < 0\}\right)$: curvature -1
    - ...bounded by $\mathrm{Ein}^{n,1} = \mathbb{P}\left(\{\mathrm{v} \in \mathbf{R}^{n+1,2} | \mathrm{v} \cdot \mathrm{v} = 0\}\right)$
    - which bounds $\mathrm{dS}^{n+1,1} = \{\mathrm{v} \in \mathbf{R}^{n+1,2} | \mathrm{v} \cdot \mathrm{v} = 1\} \cong \mathbb{P}\left(\{\mathrm{v} \in \mathbf{R}^{n,1} | \mathrm{v} \cdot \mathrm{v} > 0\}\right)$: curvature $+1$



Anti-deSitter space

- Three-dimensional models AdS
  - Hyperboloid model $\mathrm{AdS} \cong \{\mathrm{v} \in \mathbf{R}^{2,2} | \mathrm{v} \cdot \mathrm{v} = -1\} / \pm 1$
  - Projective model:
    $\mathrm{AdS} \cong \{[v_1 : v_2 : v_3 : v_4] | \mathrm{v} \cdot \mathrm{v} < 0\}$
  - Klein model
    - Projection of Hyperboloid onto $v_4 = 1$
    - $\mathrm{AdS} = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} | v_1^2 + v_2^2 - v_3^2 - 1 < 0 \right\}$
    - Misses points with $v_4 = 0$ on the hyperboloid.



$PSL(2, \mathbf{R})$ is Anti-deSitter Space

- $PSL(2, \mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | ad - bc = 1 \right\} / \pm 1$
  - det a $(2, 2)$-signature.
    - Change of variables: $a = w + y, b = x + z, c = x - z, d = w - y$
    - $ad - bc = (w + y)(w - y) - (x + z)(x - z) = 1$ or $x^2 + y^2 - z^2 - w^2 = -1$
- $\text{Isom} = PSL(2, \mathbf{R}) \times PSL(2, \mathbf{R})$
  - Preserving the quadric
  - $(A, B)X = AXB^{-1}$

Some Embedded Fun

- $f : \mathrm{PSL}(2, \mathbf{R}) \to \mathrm{AdS}^3$
  - $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [\frac{b+c}{2} : \frac{a-d}{2} : \frac{b-c}{2} : \frac{a+d}{2}] = [\frac{b+c}{a+d} : \frac{a-d}{a+d} : \frac{b-c}{a+d} : 1]$
- C is set of $A \in \mathrm{PSL}(2, \mathbf{R})$ such that attracting fixed pt. is 0 or ∞, or A has only one fixed point and it is on geodesic ℓ, where ℓ has endpint 0 and ∞.

A Crooked Plane in AsD as described by D-G-K

- $f\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = [0:0:0:1]$
- $f\left(\begin{bmatrix}1 & 2t\\ 0 & 1\end{bmatrix}\right) = [t:0:t:1]$
- $f\left(\begin{bmatrix}1 & 0\\ 2t & 1\end{bmatrix}\right) = [t:0:-t:1]$
- $f\left(\begin{bmatrix}s & t\\ 0 & 1/s\end{bmatrix}\right) = [\frac{t}{s+1/s} : \frac{s-1/s}{s+1/s} : \frac{t}{s+1/s} : 1]$ where $s > 1$
- $f\left(\begin{bmatrix}1/s & \\ t & s\end{bmatrix}\right) = [\frac{t}{s+1/s} : \frac{1/s-s}{s+1/s} : \frac{-t}{s+1/s} : 1]$ where $s > 1$
- $f\left(\begin{bmatrix}s & t\\ r & s\end{bmatrix}\right) = [\frac{r+t}{2s} : 0 : \frac{t-r}{2s} : 1]$ where $|2s| < 1$ and $s^2 - rt = 1$

References

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