An introduction to Higher Teichmüller theory: Anosov representations for rank one people

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Overview

- Classical Teichmüller theory studies the space of discrete, faithful representations of surface groups into $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}_+(\mathbb{H}^2)$.
- Many aspects of this theory can be transported into the setting of Kleinian groups (a.k.a. somewhat higher Teichmüller theory), which studies discrete, faithful representations into $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}_+(\mathbb{H}^3)$. More generally, one may study representations into isometries groups of real, complex, quaternionic or octonionic hyperbolic spaces, i.e. rank one Lie groups.
- Higher Teichmüller Theory attempts to create an analogous theory of representations of hyperbolic groups into higher rank Lie groups, e.g. $\mathrm{PSL}(n,\mathbb{R})$. Much of this theory can be expressed in the language of Anosov representations, which appear to be the correct generalization of the notion of a convex cocompact representation into a rank one Lie group.

Fuchsian representations

- Let S be a closed, oriented surface of genus at least 2.
- A representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ is **Fuchsian** if it is discrete and faithful.
- If $N_\rho = \mathbb{H}^2/\rho(\pi_1(S))$, then there exists a homotopy equivalence $h_\rho : S \to N_\rho$ (in the homotopy class of ρ).
- Baer's Theorem implies that h_ρ is homotopic to a homeomorphism.
- We may choose $x_0 \in \mathbb{H}^2$ and define the **orbit map** $\tau_\rho : \pi_1(S) \to \mathbb{H}^2$ by $\tau_\rho(g) = \rho(g)(x_0)$.
- **Crucial property 1:** If S is a closed surface and $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ is Fuchsian, then the orbit map is a quasi-isometry.

Quasi-isometries

- If Γ is a group generated by $S = \{\sigma_1, \ldots, \sigma_n\}$, we define the **word metric** on $\pi_1(S)$ by letting $d(1,\gamma)$ be the minimal length of a word in S representing γ. Then let $d(\gamma_1,\gamma_2) = d(1,\gamma_1\gamma_2^{-1})$. With this definition, each element of Γ acts by multiplication on the right as an isometry of Γ.
- A map $f : Y \to Z$ between metric spaces is a (K,C)-**quasi-isometric embedding** if

$$\frac{1}{K}d(y_1,y_2) - C \leq d(f(y_1),f(y_2)) \leq Kd(y_1,y_2) + C$$

  for all $y_1, y_2 \in Y$. The map f is a (K,C)-**quasi-isometry** if, in addition, for all $z \in Z$, there exists $y \in Y$ such that $d(f(y),Z) \leq C$.
- Basic example: The inclusion map of $\mathbb{Z}$ into $\mathbb{R}$ is a $(1,1)$-quasi-isometry, with quasi-inverse $x \to \lceil x \rceil$.

The Milnor-Svarc Lemma

Milnor-Svarc Lemma: If a group Γ acts properly and cocompactly by isometries on a complete Riemannian manifold X (or more generally on a proper geodesic metric space), then Γ is quasi-isometric to X.

Idea of proof: If Γ is generated by $S = \{\sigma_1, \ldots, \sigma_n\}$ and $x_0 \in X$, let $K = \max\{d(x_0, \sigma_i(x_0))\}$, then the orbit map $\tau : \Gamma \to X$ is K-Lipschitz.
Let C be the diameter of X/Γ, let $\{\gamma_1, \ldots, \gamma_r\}$ be the collection of elements moving x_0 a distance at most $3C$, and let $R_1 = \max\{d(1, \gamma_i)\}$, then an element $\gamma \in \Gamma$ has word length at most $R_1(\frac{d(x_0,\gamma(x_0))}{C} + 1)$.
Then the orbit map is a $(\max\{K, \frac{R_1}{C}\}, \max\{R_1, C\})$-quasi-isometry.

Crucial Property 2: If S is a closed surface of genus at least 2 and $\rho_0 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ is Fuchsian, then there exists a neighborhood U of ρ_0 in $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$ consisting entirely of Fuchsian representations.

Basic Fact: Given (K, C), there exists L and $(\hat{K}, \hat{C})$ so that if the restriction of $\tau_\rho : \pi_1(S) \to \mathbb{H}^2$ is a (K, C)-quasi-isometric embedding on a ball of radius L about id, then τ_ρ is a $(\hat{K}, \hat{C})$-quasi-isometric embedding.

Idea of Proof: Since ρ_0 is Fuchsian, its orbit map τ_{ρ_0} is a (K_0, C_0)-quasi-isometry for some (K_0, C_0). By the exercise, there exists L such that if τ_ρ is a $(2K_0, C_0 + 1)$-quasi-isometric embedding on a ball of radius L, then τ_ρ is a quasi-isometric embedding and hence ρ is discrete and faithful. But if ρ is sufficiently near ρ_0, then the orbit map τ_ρ is very close to τ_{ρ_0} on the ball of radius L and hence is a $(2K_0, C_0 + 1)$-quasi-isometric embedding on the ball of radius L.

- $$\mathcal{T}(S) \subset X(\pi_1(S), \mathrm{PSL}(2,\mathbb{R})) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

  is the space of (conjugacy classes of) Fuchsian representations ρ so that h_ρ is orientation-preserving.
- $$\mathcal{T}(\bar{S}) \subset X(\pi_1(S), \mathrm{PSL}(2,\mathbb{R})) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$$

  is the space of (conjugacy classes of) Fuchsian representations ρ so that h_ρ is orientation-reversing.
- **Basic Facts:** (See François' talks) $\mathcal{T}(S)$ is a component of $X(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$ and is homeomorphic to $\mathbb{R}^{6g-6}$.
- *Remark:* We have already seen that $\mathcal{T}(S)$ is open, by stability. It is closed, since the Margulis Lemma guarantees that a limit of discrete, faithful representations is discrete and faithful.

- Let X be a real, complex, quaternionic or octonionic hyperbolic space and let $G = \mathrm{Isom}(X)$. G is a rank one Lie group.
- If Γ is a torsion-free finitely presented group, we say that $\rho : \Gamma \to G$ is **convex cocompact** if the orbit map $\tau_\rho : \Gamma \to X$ is a quasi-isometric embedding. (Notice that we do not require that it is a quasi-isometry, so the action of $\rho(\Gamma)$ need not be cocompact.)
- A convex cocompact representation is discrete and faithful.
- If ρ_0 is convex cocompact, then there is a neighborhood of ρ_0 in $\mathrm{Hom}(\Gamma, G)$ consisting of convex cocompact representations.

Cautionary Tales

- The set $CC(\Gamma, G)$ of convex cocompact representations need not be a collection of components of $\mathrm{Hom}(\Gamma, G)$. For example, a convex cocompact representation of the free group into G may always be deformed to the trivial representation (and there are always convex cocompact representations of the free group into G.)
- Discrete, faithful representations need not be convex cocompact. Abstractly, this follows from the previous statement, since the set of discrete, faithful represention*s* is closed in $\mathrm{Hom}(\Gamma, G)$. More concretely, consider the limit of Schottky groups in $\mathrm{PSL}(2, \mathbb{R})$ where the circles are allowed to touch. These are discrete, faithful representations with parabolic elements in their image.
- (Brock-Canary-Minsky,Bromberg,Magid) The set $DF(\pi_1(S), \mathrm{PSL}(2, \mathbb{C}))$ is the closure of $CC(\pi_1(S), \mathrm{PSL}(2, \mathbb{C}))$, but is not locally connected.

The bending construction

- Suppose that a curve C cuts a surface S of genus at least 2 into two pieces S_0 and S_1, so that

$$\pi_1(S) = \pi_1(S_0) *_{\pi_1(C)} \pi_1(S_1).$$

Let $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian representation.

- Given θ, one may construct a representation $\rho_\theta : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ by constructing a map of $\mathbb{H}^2$ into $\mathbb{H}^3$ by iteratively bending ρ by an angle θ along pre-images of the geodesic representative of C on $N_\rho = \mathbb{H}^2/\rho(\pi_1(S))$.
- Algebraically, let L be the axis of each non-trivial element of $\rho(\pi_1(C))$ and let $R_\theta \in \mathrm{PSL}(2,\mathbb{C})$ be the rotation of angle θ in L. Let $\rho_\theta = \rho$ on $\pi_1(S_0)$ and let $\rho_\theta = R_\theta \rho R_\theta^{-1}$ on $\pi_1(S_1)$.
- By stability, ρ_θ will be convex cocompact for small θ, but ρ_π will not be discrete and faithful (since it is a Fuchsian representation with volume 0).

Limit maps

- A proper geodesic metric space X is **hyperbolic** if there exists δ so that any geodesic triangle is δ-thin (i.e. any side lies in the δ-neighborhood of the other two sides).
- If X is hyperbolic $\partial_\infty X$ is the set of (equivalence classes of) geodesic rays so that two geodesic rays which remain within a bounded neighborhood of one another are regarded as equivalent.
- A group Γ is **word hyperbolic** if its Cayley graph C_Γ is hyperbolic. Let $\partial_\infty \Gamma = \partial_\infty C_\Gamma$.
- A quasi-isometric embedding $f : X \to Y$ extends to an embedding $\hat{f} : \partial_\infty X \to \partial_\infty Y$.
- **Crucial Property 3:** If Γ is hyperbolic, $G = \mathrm{Isom}_+(X)$ is a rank one Lie group, and $\rho : \Gamma \to G$ is convex cocompact, then we get an embedding $\xi_\rho : \partial_\infty \Gamma \to \partial_\infty X$ which is called the **limit map**.

What is Higher Teichmüller theory

- **Goal:** Construct a theory of representations of a hyperbolic group Γ into an arbitrary semi-simple Lie group G, e.g. $PSL(n,\mathbb{R})$, which captures some of the richness of Teichmüller theory.
- **Question:** Why not just use the earlier definition?
- **Problem:** The symmetric space $X = G/K$ associated to G is only non-positively curved, not negatively curved. If G has rank n, then X contains totally geodesic copies of Euclidean space $\mathbb{E}^n$.
- Notice that a sequence of rotations in $\mathbb{E}^2$ can converge to a translation in $\mathbb{E}^2$, so "quasi-isometric embedding are not stable."
- (Guichard) There exists a representation $\rho : F_2 \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ such that the associated limit map $\tau_\rho : F_2 \to \mathbb{H}^2 \times \mathbb{H}^2$ is a quasi-isometric embedding, yet ρ is a limit of non-faithful representations.

Projective Bending

- Suppose that $\rho_0 : \pi_1(S) \to \mathrm{Isom}_+(\mathbb{H}^2) = SO(2,1) \subset \mathrm{PSL}(3,\mathbb{R})$ is Fuchsian and again $\pi_1(S) = \pi_1(S_0) *_{\pi_1(C)} \pi_1(S_1)$. Choose a one-parameter family $\{R_t\}$ in $Z(\pi_1(C)) \subset \mathrm{PSL}(3,\mathbb{R})$ so that $R_0 = I$.
- Let $\rho_t = \rho$ on $\pi_1(S_0)$ and $\rho_t = R_t \rho_0 R_t^{-1}$ on $\pi_1(S_1)$.
- One may generalize this construction when $\rho_0 : \Gamma \to \mathrm{Isom}_+(\mathbb{H}^n) = SO(n,1) \subset \mathrm{PSL}(n+1,\mathbb{R})$ is cocompact and there exists an embedded totally geodesic codimension one submanifold of $\mathbb{H}^n/\rho(\Gamma)$.
- (Benoist) **Each** of these deformations is the holonomy of a convex projective structure on the surface (or manifold), i.e. $\rho_t(\Gamma)$ preserves and acts properly discontinuously on a strictly convex domain Ω_t in $\mathbb{RP}^n$ with C^1 boundary.
- We call the component of $X(\Gamma, \mathrm{PSL}(n+1,\mathbb{R}) = \mathrm{Hom}(\Gamma, \mathrm{PSL}(n+1,\mathbb{R}))/\mathrm{PSL}(n+1,\mathbb{R})$ containing ρ_0 a *Benoist component*.

The irreducible representation

- We recall the irreducible representation

$$\tau_n : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R}).$$

- Regard $\mathbb{R}^n$ as the vector space of degree $n-1$ homogeneous polynomials in 2 variables, i.e.

$$\mathbb{R}^n = \{a_1 x^{n-1} + a_2 x^{n-2} y + \cdots + a_n y^{n-1}\}.$$

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\tau_n(A)$ acts on $\mathbb{R}^n$ by taking x to $ax+by$ and taking y to $cx+dy$.
- For example, if $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, then

$$\tau_n(A)\left(a_1 x^{n-1} + a_2 x^{n-2} y + \cdots + a_n y^{n-1}\right)$$

$$= a_1 \lambda^{n-1} x^{n-1} + a_2 \lambda^{n-3} x^{n-2} y + \cdots + a_n \lambda^{1-n} y^{n-1}.$$

The irreducible representation II

- In other words,

$$\tau_n\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\right) = \begin{bmatrix}\lambda^{n-1} & 0 & \cdots & 0\\ 0 & \lambda^{n-3} & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda^{1-n}\end{bmatrix}$$

- Notice that $\tau_n\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\right)$ is diagonalizable with distinct eigenvalues.

The Hitchin component

- The irreducible representation induces an embedding
$$\mathcal{T}(S) \to \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R})$$
given by taking ρ to $\tau_n \circ \rho$.
- The component $H_n(S)$ of
$$X(\pi_1(S), PSL(n,\mathbb{R})) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R})$$
which contains the image of $\mathcal{T}(S)$ is called the **Hitchin component**.
- The image of $\mathcal{T}(S)$ is called the **Fuchsian locus.**
- Hitchin showed that $H_n(S)$ is an analytic manifold diffeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$ and called it the Teichmüller component.
- $H_3(S)$ is also a Benoist component.

Geodesic Flows

- For simplicity, let $\Gamma = \pi_1(M)$ where M is a closed negatively curved manifold and let U_Γ denote the geodesic flow on T^1M. Notice that Γ is a hyperbolic group since it is quasi-isometric to the hyperbolic metric space $\tilde{M}$.
- One may consider $\widetilde{U_\Gamma} = T^1\tilde{M}$ and identify

$$\widetilde{U_\Gamma} = (\partial_\infty \tilde{M} \times \partial_\infty \tilde{M} - \Delta) \times \mathbb{R} = (\partial_\infty \Gamma \times \partial_\infty \Gamma - \Delta) \times \mathbb{R}.$$

- The geodesic flow of a closed negatively curved manifold is *Anosov*, i.e. the tangent space of T^1M at any point splits as $V_+ \oplus N \oplus V_-$ where E is a line in the direction of the flow and, infinitesmally, the flow is expanding on V_+ and contracting on V_-.
- If $\tilde{M} = \mathbb{H}^2$ and $v \in T^1\mathbb{H}^2$, then let $\tilde{L}^-$ be the curve in $T^1\mathbb{H}^2$ obtained by moving v along the horocycle perpendicular to the geodesic through v which passes through the positive endpoint of the geodesic γ_v in the direction of v. Then $\tilde{V}^-|_v$ is the tangent space to $\tilde{L}^-$.

Transverse limit maps

- A representation $\rho : \Gamma \to \mathrm{PSL}(n, \mathbb{R})$ has **transverse limit maps** if there exist ρ-equivariant maps

$$\xi_\rho : \partial_\infty \Gamma \to \mathbb{RP}^{n-1} \quad \text{and} \quad \theta_\rho : \partial_\infty \Gamma \to \mathrm{Gr}_{n-1}(\mathbb{R}^n)$$

  so that if $x \neq y \in \partial_\infty \Gamma$, then $\xi_\rho(x) \oplus \theta_\rho(y) = \mathbb{R}^n$.
- If we assume, in addition, that $\xi_\rho(\partial\Gamma)$ spans $\mathbb{R}^n$, then $\rho(\gamma^+)$ is an attracting eigenline for $\rho(\gamma)$ and $\rho(\gamma^-)$ is a repelling hyperplane for $\rho(\gamma)$, so $\rho(\gamma)$ is **proximal**, i.e. has a real eigenvalue of maximal modulus and multiplicity 1. In fact, $\rho(\gamma)$ is **biproximal**, i.e. its inverse is also proximal.
- It follows that in this case the limit maps are unique, if they exist. Notice that our assumption is equivalent to the assumption that $\rho(\Gamma)$ is irreducible, i.e. preserves no proper vector subspace.
- (Guichard-Wienhard) An irreducible representation with transverse limit maps is projective Anosov.

Flat bundles and their splittings

- Let E_ρ be the flat bundle over U_Γ determined by ρ, i.e. let

$$\tilde{E}_\rho = \tilde{U}_\Gamma \times \mathbb{R}^n$$

  and let Γ act on $\tilde{E}_\rho$ as the group of covering transformations of U_Γ in the first factor and as $\rho(\Gamma)$ in the second factor, then let

$$E_\rho = \tilde{E}_\rho / \Gamma.$$

- If ρ has transverse limit maps ξ_ρ and θ_ρ, one gets an equivariant splitting

$$\tilde{E}_\rho = \tilde{\Xi} \oplus \tilde{\Theta}$$

  where $\tilde{\Xi}$ is the line bundle over $\tilde{U}_\Gamma$ whose fiber over (x, y, t) is the line $\xi_\rho(x)$ and $\tilde{\Theta}$ is the hyperplane bundle over $\tilde{U}_\Gamma$ whose fiber over (x, y, t) is the hyperplane $\theta_\rho(y)$.
- This descends to a splitting

$$E_\rho = \Xi \oplus \Theta.$$

Projective Anosov representations

- The geodesic flow $\{\phi_t\}$ on U_Γ lifts to a flow $\{\hat{\phi}_t\}$ on E_ρ parallel to the flat connection. Explicitly, let $\{\tilde{\phi}_t\}$ be the lift of $\{\phi_t\}$ to $\tilde{U}_\Gamma$, then we get a "stupid" flow $\{\tilde{\phi}'_t\}$ on $\tilde{E}_\rho$ given by
$$\tilde{\phi}'_t((x,y,s),v) = ((x,y,s+t,v) = (\tilde{\phi}_t(x,y,s),v)$$
which descends to a flow $\{\hat{\phi}_t\}$ on E_ρ
- Notice that this flow preserves the splitting by construction.
- A representation $\rho : \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ with transverse limit maps is **Projective Anosov** if the flow is contracting on $\mathrm{Hom}(\Theta,\Xi) = \Xi \otimes \Theta^*$. This implies, by abstract nonsense, that the flow is contracting on Ξ.
- Since $\xi_\rho(\gamma^+)$ is preserved by $\rho(\gamma)$, it is an eigenline. The fact that the flow is contracting on Ξ implies that the modulus $\lambda_1(\rho)$ of the associated eigenvalue is bigger than 1. Since U_Γ is compact, the flow is uniformly contracting which implies that $\log \lambda_1$ is comparable to the length of the period in U_Γ associated to Γ.

Well-displacing representations

The length of the period in U_Γ associated to γ is comparable to the reduced word length $||\gamma||$ of γ, so there exist $K > 0$ and $C > 0$ such that

$$\log(\lambda_1(\rho(\gamma)) \geq K||\gamma|| - C$$

for all $\gamma \in \Gamma$. We say that ρ is **well-displacing**.

We have the following more precise contraction property.

Lemma: A representation with transverse limit maps is projective Anosov if and only if given any continuous norm on E_ρ, there exists $t_0 > 0$ such that if $Z \in U_\Gamma$, $v \in \Xi_Z$ and $w \in \Theta_Z$, then

$$\frac{||\hat{\phi}_{t_0}(v)||_{\hat{\phi}_{t_0}(Z)}}{||\hat{\phi}_{t_0}(w)||_{\hat{\phi}_{t_0}(Z)}} \leq \frac{1}{2}\frac{||v||_Z}{||w||_Z}.$$

One upshot of this is that $\lambda_1(\rho(\gamma))$ is the eigenvalue of maximal modulus.

Quasi-isometric embedding

- (Delzant, Guichard, Labourie, Mozes) A well-displacing representation has an orbit map which is a quasi-isometric embedding.
- Basic idea: A linear lower bound on $\log \lambda_1(\rho(\gamma))$ in terms of $||\gamma||$ provides a linear lower bound of the translation length of $\rho(\gamma)$ on the associated symmetric space.
- Since ρ is a quasi-isometric embedding and Γ is torsion-free, ρ is discrete and faithful.
- **Summary:** Projective Anosov representations are discrete, faithful and well-displacing, the associated orbit map is a quasi-isometric embedding and the image of every element is biproximal.

Examples

- (Benoist) Representations in a Benoist component are projective Anosov. In the examples we discussed, one identifies $\partial\Omega_t$ with $\partial_\infty\Gamma$ which gives the map $\xi_{\rho_t} : \partial_\infty\Gamma \to \mathbb{RP}^n$. One then lets $\theta_{\rho_t}(x)$ be the tangent plane to Ω_t at $\xi_\rho(x)$. The transversality is a consequence of the strict convexity of Ω_t. Each ρ_t is irreducible, so this is enough to guarantee that ρ_t is projective Anosov.
- (Labourie) Hitchin representations are projective Anosov. In fact, they are Anosov with respect to a minimal parabolic subgroup B and there is a limit map $\hat{\xi}_\rho : \partial_\infty\pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})/B = \mathrm{Flag}(\mathbb{R}^n)$ and ξ_ρ and θ_ρ are obtained by projecting onto a factor.
- (Benoist, Quint) If $\{\gamma_1, \ldots, \gamma_n\}$ is a finite collection of proximal elements in general position, then $\rho_r : F_n \to \mathrm{PSL}(n,\mathbb{R})$ given by $\rho_n(x_i) = \gamma_i^r$ is projective Anosov for all large enough r.

Anosov Representations

- Let G be a semi-simple Lie group and $P^{\pm}$ a pair of opposite parabolics. For example, $G = \mathrm{PSL}(n, \mathbb{R})$, P^+ the stabilizer of a partial flag and P^- the stabilizer of the dual partial flag.
- Let $L = P^+ \cap P^-$ be the Levi subgroup. $\mathcal{X} = G/L$ is an open subset of $G/P^+ \times G/P^-$.
- Given a representation $\rho : \Gamma \to G$ we form $\tilde{\mathcal{X}}_\rho = \tilde{U}_\Gamma \times \mathcal{X}$ and let Γ act by covering transformations on the first factor and by $\rho(\Gamma)$ on the second factor. Let $\mathcal{X}_\rho = \tilde{\mathcal{X}}_\rho / \Gamma$.
- Given a section $\sigma : U_\Gamma \to \mathcal{X}_\rho$, then the lift $\tilde{\sigma} : \tilde{U}_\Gamma \to \tilde{\mathcal{X}}_\rho$ splits as $\tilde{\sigma} = (id, \tilde{\sigma}_+, \tilde{\sigma}_-)$. We get associated bundles $\tilde{N}_\rho^+$ and $\tilde{N}_\rho^-$ over U_Γ so that the fiber of $\tilde{N}_\rho^\pm$ over $Z \in U_\Gamma$ is $T_{\sigma^\pm(Z)} G/P^\pm$.
- The geodesic flow again lifts in a trivial manner to a geodesic flow on $\tilde{N}_\rho^\pm$ and descends to the quotient $N_\rho^\pm = \tilde{N}_\rho^\pm / \Gamma$.

Anosov sections

σ is an **Anosov section** if

1. $\tilde{\sigma}^{\pm}(x, y, t)$ depends only on x and y (i.e. σ is flat along orbits of the flow)
2. The geodesic flow is contracting on N_ρ^+.
3. The geodesic flow is expanding on N_ρ^-.

(Labourie) A representation ρ is **Anosov** with respect to $P^{\pm}$ if $\tilde{X}_\rho$ admits an Anosov section.

The contracting/expanding properties imply that σ^+ depends only on x and σ^- depends only on y, so we get limit maps $\xi_\rho^{\pm} : \partial_\infty \Gamma \to G/P^{\pm}$. They are transverse since $\tilde{\sigma}$ has image in $\mathcal{X}$.

Examples

- If G has rank 1, there is a unique pair of opposite parabolic subgroups (up to conjugacy) and ρ is Anosov if and only if ρ is convex cocompact.
- Projective Anosov representation are Anosov with respect to $P^{\pm}$, where P^{+} is the stabilizer of a line and P^{-} is the stabilizer of a complementary hyperplane.
- Hitchin representations are Anosov with respect to $P^{\pm}$ where P^{+} is the stabilizer of a flag, i.e. the set of upper triangular matrices (up to conjugacy).
- (Guichard-Wienhard) Given any G and $P^{\pm}$, there is an irreducible representation $\eta : G \to \mathrm{PSL}(n, \mathbb{R})$, called a Plücker embedding, such that $\rho : \Gamma \to G$ is Anosov with respect to $P^{\pm}$ if and only if $\eta \circ \rho$ is projective Anosov.

Theorem: (Labourie, Guichard-Wienhard) If $\rho : \Gamma \to G$ is Anosov with respect to $P^{\pm}$, then

1. ρ is discrete and faithful,
2. $\tau_\rho : \Gamma \to G/K$ is a quasi-isometric embedding,
3. $\rho(\gamma)$ is proximal with respect to $P^{\pm}$.
4. There is a neighborhood U of ρ in $\mathrm{Hom}(\Gamma, G)$ consisting of representations which are Anosov with respect to $P^{\pm}$.
5. $\mathrm{Out}(\Gamma)$ acts properly discontinuously on the space $\mathrm{Anosov}(\Gamma, G)$ of (conjugacy classes) of Anosov representations of Γ into G.

For example, if $\rho : \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$ is Hitchin, then $\rho(\gamma)$ is diagonalizable over $\mathbb{R}$ with distinct eigenvalues.

Theorem: (Bridgeman-C-Labourie-Samborino) The limit maps ξ_ρ vary analytically as ρ varies analytically.

Other definitions

Both Gueritaud-Guichard-Kassel-Wienhard and Kapovich-Leeb-Porti now have definitions which avoid the consideration of flow spaces. Both also have definitions which avoid even the limit map.

To oversimplify, the work of GGKW involves a study of the Cartan projection, while the work of KLP involves studying the action of the group on the symmetric space and its boundaries.