Pressure Metrics for Higher Teichmüller spaces

Dick Canary
(joint work with Martin Bridgeman, François Labourie and Andres Sambarino)

University of Michigan

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Thurston's Riemannian metric on Teichmüller space

- If $T > 0$ and $\rho \in \mathcal{T}(S)$, let
$$R_\rho(T) = \{[g] \in [\pi_1(S)] \mid \ell(\rho(g)) \leq T\}$$
where $\ell(\rho(g))$ is the translation length of $\rho(g)$. Notice that $\ell(\rho(g)) = 2\log(\lambda_1(\rho(g)))$.
- If ρ is Fuchsian, then $h(\rho) = \lim \frac{\log \#(R_\rho(T))}{T} = 1$.
- If $\rho_1, \rho_2 \in \mathcal{T}(S)$, we define their intersection
$$I(\rho_1, \rho_2) = \lim_{T\to\infty} \frac{1}{\#(R_{\rho_1}(T))} \sum_{[g]\in R_{\rho_1}(T)} \frac{\ell(\rho_2(g))}{\ell(\rho_1(g))}.$$
- We may interpret this as the length in X_σ of a random unit length geodesic X_ρ.
- One then defines $I_\rho : \mathcal{T}(S) \to \mathbb{R}$ by $I_\rho(\sigma) = I(\rho, \sigma)$. Thurston proved that I_ρ has a minimum at ρ and that the Hessian of I_ρ is positive definite.
- Wolpert proved that the resulting metric on $\mathcal{T}(S)$ is a multiple of the Weil-Petersson metric.

- Let $\{\rho_u\}_{u\in M}$ be a family of (conjugacy classes of) projective Anosov representations $\rho_u : \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ parameterized by an analytic manifold M. We say that $\{\rho_u\}_{u\in M}$ is an **analytic family** of projective Anosov representations.
- We say that ρ_u is **generic** if there exists $\gamma \in \Gamma$, so that $\rho_u(\gamma)$ is diagonalizable over $\mathbb{C}$ with distinct eigenvalues.
- Labourie proved that every point in $H_n(S)$ is irreducible and generic.
- Johnson and Millson observed that Benoist components are not smooth in general, but every representation in a Benoist component is irreducible and generic.

Entropy and intersection number

- If $T > 0$ and $\rho : \Gamma \to \mathrm{PSL}(n, \mathbb{R})$ is projective Anosov, let

$$R_\rho(T) = \{[\gamma] \in [\Gamma] \mid \log(\lambda_1(\rho(\gamma)) \le T\}.$$

- The **entropy** of ρ is defined to be

$$h(\rho) = \lim_{T\to\infty} \frac{\log(\#(R_\rho(T))}{T}.$$

- If $\rho_1, \rho_2 \in \mathrm{Hom}(\Gamma, G)$ are projective Anosov, we define their **intersection number**

$$I(\rho_1, \rho_2) = \lim_{T\to\infty} \frac{1}{\#(R_{\rho_1}(T))} \sum_{[\gamma]\in R_{\rho_1}(T)} \frac{\log(\lambda_1(\rho_2(g)))}{\log(\lambda_1(\rho_1(g)))}.$$

- We define their **renormalized intersection number** to be

$$J(\rho_1, \rho_2) = \frac{h_{\rho_2}}{h_{\rho_1}} I(\rho_1, \rho_2).$$

Analyticity

Theorem:(BCLS) If $\{\rho_u\}_{u\in M}$ is an analytic family of projective Anosov representations, then the entropy $h(\rho_u)$ varies analytically over M and the intersection number $I(\rho_u, \rho_v)$ and renormalized intersection number $J((\rho_u, \rho_v)$ vary analytically over $M \times M$.

Moreover, if $u \in M$, then $J_u : M \to \mathbb{R}$ given by $J_u(v) = J(\rho_u, \rho_v)$ achieves its minimum at u.

Historical Remarks: Burger first defined a renormalized intersection number for convex cocompact representations and showed, in this setting, that J_u achieves its minimum at u. Ruelle showed analyticity of entropy on quasifuchsian space $CC(\pi_1(S), \mathrm{PSL}(2,\mathbb{C}))$ and Schottky space $CC(F_n, \mathrm{PSL}(2,\mathbb{C}))$. Anderson-Rocha generalized Ruelle's work to the setting of free products of surface groups. Pollicott-Sharp showed analyticity of a related entropy function on the Hitchin component.

The pressure form and metric

The **pressure form** on $T_u M$ is simply the Hessian of J_u. It follows from the above theorem that the pressure form is analytic and non-negative.

Theorem:(BCLS) The pressure form is positive definite at all irreducible, generic points on M.

Corollary: (BCLS) The pressure form is an analytic Riemannian metric on $H_n(S)$ which is invariant under the action of the mapping class group and restricts to a multiple of the Weil-Petersson metric on the Fuchsian locus.

The invariance under the action of the mapping class group follows immediately from the definition while the restriction to the Fuchsian locus is given by Wolpert's result.

Other deformation spaces

- Let $\mathcal{G}(\Gamma, n) \subset X(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$ denote the set of smooth points which are irreducible and generic representations.
- **Corollary:** The pressure form is an analytic Riemannian metric on $\mathcal{G}(\Gamma, n)$ which is invariant under the action of $\mathrm{Out}(\Gamma)$.
- Let G be a semi-simple Lie group and $P^{\pm}$ a pair of opposite parabolic subgroups. Let $\mathcal{Z}(\Gamma, G, P^{\pm}) \subset X(\Gamma, G)$ denote the set of smooth points which are (virtually) Zariski dense representations.
- **Corollary:** The pressure form is an analytic Riemannian metric on $\mathcal{Z}(\Gamma, G, P^{\pm})$ which is invariant under the action of $\mathrm{Out}(\Gamma)$.

Deformation spaces of Kleinian groups

- $\rho \in CC(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ is Zariski dense, unless Γ is free or a surface group and $\rho(\Gamma)$ is (almost) conjugate into $\mathrm{PSL}(2,\mathbb{R})$.
- **Corollary:** The pressure form is an $\mathrm{Out}(\Gamma)$ invariant, analytic, Riemannian metric on $CC(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ if Γ is not free or a surface group. In all cases, it gives rise to a path metric.
- (Patterson,Sullivan) If $\rho \in CC(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$, $h(\rho)$ is the Hausdorff dimension of the limit set.
- **Corollary:** The Hausdorff dimension of the limit set varies analytically over $CC(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$.
- More generally, if G is a rank one Lie group, one may use work of Corlette-Iozzi and Yue to show that the Hausdorff dimension of the limit set varies analytically over analytic families of convex cocompact representations of Γ into G.

Historical Remarks

(1) Bonahon first gave a formulation of the Weil-Peterson metric on Teichmüller space in terms of intersection number. McMullen first gave an interpretation of the Weil-Peterson metric on Teichmüller space in terms of the Thermodynamic formalism.
(2) Motivated by McMullen's work, Bridgeman developed a pressure metric on quasifuchsian space which is an analytic Riemannian metric off of the Fuchsian locus and restricts to a multiple of the Weil-Petersson metric on the Fuchsian locus.
(3) When $n = 3$, Qiongling Li constructed a metric on $H_3(S)$, called the Loftin metric, with the same properties. An earlier metric was constructed by Davishzadeh-Goldman and Li shows that this metric also shares these properties.

Outline of Proof

1. Given a projective Anosov representation ρ find a metric Anosov flow U_ρ which is a reparameterization of U_Γ by a positive Hölder function f_ρ and encodes the spectral radii of elements of $\rho(\Gamma)$. (Actually, f_ρ is only defined up to Liv̌sic cohomology.)
2. Given an analytic family $\{\rho_u\}_{u\in M}$, show that f_{ρ_u} varies analytically in the space $\mathcal{H}$ of Hölder functions on U_Γ.
3. Our analyticity theorem then follows immediately from the thermodynamic formalism.
4. We define a **Thermodynamic mapping** $u \to -h(\rho_u)f_u$ into the space $\mathcal{P}(U_\Gamma)$ of of pressure zero Hölder functions on U_Γ which is well-defined locally and well-defined globally up to Liv̌sic cohomology. Our pressure form is a pullback of a pressure form on $\mathcal{P}(U_\Gamma)$.
5. The thermodynamic formalism gives rise to a criterion for non-degeneracy of the pressure form at a point, which we have to work hard to establish.

The geodesic flow of a projective Anosov representation

- The idea for the construction of U_ρ originates in work of Andres Sambarino.
- Let F_ρ be the $\mathbb{R}$-bundle over $\partial_\infty\Gamma \times \partial_\infty\Gamma - \Delta$ such that the fiber over a point (x, y) is the space of vectors in $\xi_\rho(x)$ (where $v \sim -v$).
- There is flow $\{\tilde{\psi}_t\}$ on F_ρ where

$$\tilde{\psi}_t(x, y, u) = (x, y, e^t v).$$

- Γ acts on F_ρ by acting on the base by the usual action and on the fiber by $\rho(\Gamma)$
- **Key Fact:** There exists an equivariant Hölder orbit equivalence $\tilde{g}_\rho : \tilde{U}_\Gamma \to F_\rho$.
- It follows that Γ acts properly discontinuously and cocompactly on F_ρ, the flow $\{\tilde{\psi}_t\}$ on F_ρ descends to a flow ψ_t on the quotient $U_\rho = F_\rho/\Gamma$, and $\tilde{g}_\rho$ descends to a Hölder orbit equivalence $g_\rho : U_\Gamma \to U_\rho$.

More on the geodesic flow

- If we know that U_Γ is Anosov, it follows that U_ρ is metric Anosov. If not, we can prove directly that U_ρ is metric Anosov.
- It follows from the general theory of Anosov flows that we may find a positive Hölder function f_ρ so that U_ρ is a reparameterization of U_Γ by f_ρ. (In fact, to establish analytic variation of f_ρ we will essentially find a formula for f_ρ.)
- Notice that the orbit of U_ρ associated to γ is the quotient of the fiber of F_ρ over (γ^+, γ^-). Notice that γ identifies (γ^+, γ^-, v) with

$$(\gamma^+, \gamma^-, \lambda_1(\rho(\gamma)v) = \phi_{\log \lambda_1(\rho(\gamma))}(\gamma^+, \gamma^-, v).$$

  So, the orbit has period $\log \lambda_1(\rho(\gamma))$.
- Therefore, $h(\rho)$ is the topological entropy of U_ρ, so the name entropy made sense.

Pressure and Entropy

- If $\phi = \{\phi_t\}$ is a metric Anosov flow on a compact metric space X, and $f : X \to \mathbb{R}$ is a Hölder function, then $\ell_f(a)$ is the integral of f along a closed orbit a. If f is positive, then $\ell_f(a)$ is the period of the orbit a in the reparameterization ϕ^f of ϕ by f.
- We define the *Pressure* function

$$P(f) = \lim_{T\to\infty} \frac{1}{T} \log \left(\sum_{a \in R_T(\phi)} e^{\ell_f(\gamma)} \right)$$

where $R_T(\phi)$ is the set of periods of length at most T.
- P is analytic and if f is positive, then $P(-hf) = 0$ if and only if h is the topological entropy of ϕ^f.
- In our setting, $P(-h(\rho_u)f_{\rho_u}) = 0$ and we define a Thermodynamic mapping $\Theta : M \to \mathcal{P}(U_\Gamma)$ given by $u \to -h(\rho)f_{\rho_u}$ where $\mathcal{P}(U_\Gamma)$ is the space of pressure zero functions on U_Γ.

The pressure form on $\mathcal{P}(U_\Gamma)$

In analogy with a construction of McMullen in the shift space setting, we define a **pressure semi-norm** on $T_f\mathcal{P}(U_\Gamma)$ by letting

$$||g||^2 = \frac{\partial}{\partial t^2}\bigg|_{t=0} P(f+tg)$$

for all $g \in T_f\mathcal{P}(U_\Gamma)$.

It is an exercise in the Thermodynamic formalism to show that our pressure norm is the pullback of the pressure form on $\mathcal{P}(U_\Gamma)$ and that it is non-negative as claimed.

Degeneracy Criterion: A vector $v \in TM$ is degenerate for the pressure norm if and only if

$$D_v(h(f_u)\log(\lambda_1(\rho_u(\gamma))) = 0$$

for all $\gamma \in \Gamma$.

Formalism

The reparameterization function is only well-defined up to Livsic cohomology, so the Thermodynamic mapping really maps into the space of Liv̌sic cohomology classes of pressure-zero functions on U_Γ. We must show that it admits local lifts into $\mathcal{P}(U_\Gamma)$.

We should really be working in the space of α-Hölder functions which is a Banach space. Again this is only true locally, i.e. for any $u_0 \in M$ there is a neighborhood U such that f_{ρ_u} is α-Hölder for all $u \in U$.