z-Classes in Geometry and Groups

Ravindra S. Kulkarni
Bhaskaracharya Pratishthana, Pune

Advanced School and Workshop on Geometry of Discrete Actions
August 24 - September 4, 2015
ICTP, Trieste, Italy

August 31, 2015

Table of contents

Dynamical Types in Euclidean Geometry

Let G_1 be the group of orientation preserving isometries of the Euclidean plane $\mathbb{E}^2$. It consists of

- Identity,
- Translations,
- Rotations around points of $\mathbb{E}^2$.

We say that there are three "dynamical types" of elements in G_1.

Dynamical Types in Spherical Geometry

Even more simply, let G_2 be the group of orientation preserving isometries of the 2-sphere $\mathbb{S}^2$, considered as the unit sphere in $\mathbb{R}^3$. It consists of

- Identity,
- Rotations around axes passing through the origin in $\mathbb{R}^3$, with angle of rotation in $(0, \pi)$.
- Rotations around axes passing through the origin in $\mathbb{R}^3$, with angle of rotation equal to π.

We say that there are three "dynamical types" of transformations in G_2.

Dynamical Types in Hyperbolic Geomrtry

Similarly, let $\mathbb{H}^2$ be the hyperbolic plane, and $\partial\mathbb{H}^2$ be its ideal boundary. Let G_3 be the group of orientation preserving isometries of $\mathbb{H}^2$. It consists of

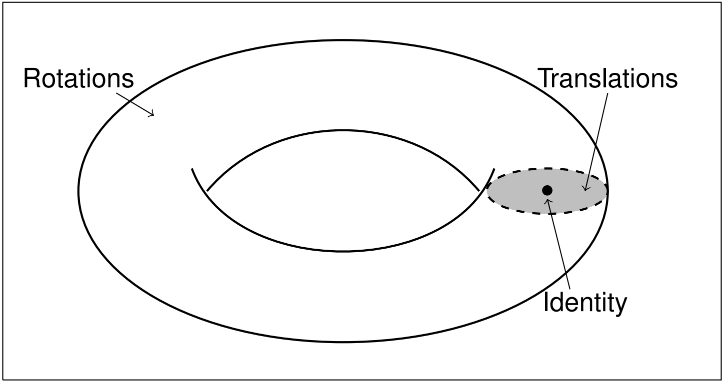
- Identity,
- Elliptics, i.e. rotations around points in $\mathbb{H}^2$,
- Hyperbolics, i.e. translations fixing two points on $\partial\mathbb{H}^2$,
- Parabolics, i.e. translations fixing one point on $\partial\mathbb{H}^2$.

We say that there are four "dynamical types" of elements in G_3.

In these three examples there are nice pictures associated with them, which, in my opinion, should be a part of a common equipment of a graduate student.

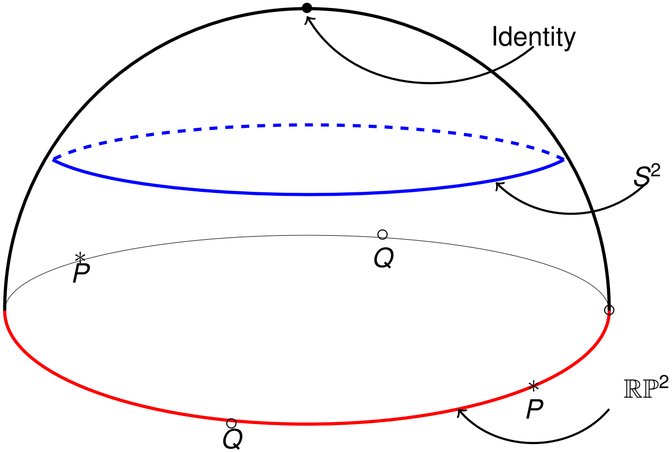
Orientation preserving isometries of Euclidean plane $\mathbb{E}^2$:

$$G_1 = \{z \mapsto az + b \colon |a| = 1, a, b \in \mathbb{C}\} \approx S^1 \times \mathbb{C} \approx S^1 \times \mathbb{D}^2$$



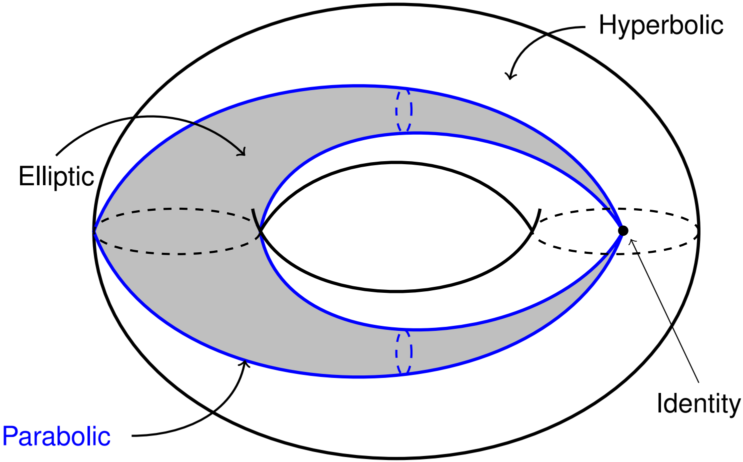
Orientation preserving isometries of Sphere S^2

$$G_2 = SO(3) \cong \mathbb{RP}^3$$



Orientation preserving isometries of Hyperbolic Plane $\mathbb{H}^2$

$$G_3 = \mathrm{PSL}(2,\mathbb{R}) \approx S^1 \times \mathbb{R}^2 \approx S^1 \times \mathbb{D}^2$$



- Similarly, in all the classical geometries defined over reals, complex numbers, or quaternions we can talk about *dynamical types*.
- Remarkably, although the groups are infinite, there are only finitely many dynamical types.
- Moreover, we also observe that each transformation has a unique spatial invariant and a unique numerical invariant.
- **Example:** A rotation of $\mathbb{E}^2$ has a unique fixed point which is its spatial invariant and the angle of rotation which is its numerical invariant.

Problem:

- What should we mean by "dynamical type" of an element, and its spatial and numerical invariants?
- Can we give a definition of a dynamical type? Preferably, in terms of group alone, independent of its action on a particular space?

z-Classes

Definition

Let G be a group. Two elements x, y in G are said to be z-equivalent if their centralisers are conjugate.

In the three examples, a computation of centralisers of elements shows that

$$z\text{-classes} \quad \longleftrightarrow \quad \text{dynamical types}$$

In the more general, higher dimensional, classical examples, one sees that the correspondence is not quite bijective, but still, finite to finite.

In particular, finiteness of the vague notion of dynamical types in a classical homogeneous geometry can be explained if we can show finiteness of z-classes in the corresponding automorphism groups.

Broad Results

1. "Spatial" and "numerical" parts of an element can be expressed by a general result on z-classes, cf. the next section.
2. Finiteness of z-classes in classical geometries over reals, complex numbers, and quaternions, can be explained by a general result on reductive Lie groups.

For compact Lie groups this is implicit in Weyl's work by now well known structure theory of compact Lie groups. In fact, as it turns out, the crucial part of Weyl's analysis is purely group-theoretical. For example, the notion of a "Weyl group" has dynamic origin.

If we consider the analogues of classical geometries over fields with richer arithmetic, such as the field $\mathbb{Q}$ of rational numbers, then there arise new arithmetic invariants of z-classes and strict finiteness does not hold. The following is a typical result.

Theorem

Let a field F have the property: there exist only finitely many field extensions of a fixed degree. Then $GL_n(F)$ has only finitely many z-classes.

cf. the recent papers by [Kulkarni], [Gongopadhyay], [Singh] for linear, orthogonal and symplectic groups, and the Lie group G_2.

Of course the interest in these papers is not just to show the finiteness result, but actually give the number of z-classes, and compute the "numerical" and "spatial" invariants of an element in each z-class.

Orbit-Classes

Definition

Let G be a group acting on a set X. We say that x, y in X are in the same *orbit class* if the stabilizers of x, y are conjugate in G.

$$\begin{aligned} G_x &= \text{the stabiliser at } x \in X \\ G(x) &= \text{the orbit of } x \in X \\ &= \{y \in X : \textit{for some } g \in G, y = G_x\}. \\ R(x) &= \text{the orbit-class of } x \in X \\ &= \{y \in X : \textit{for some } g \in G, gG_y g^{-1} = G_x\}. \end{aligned}$$

- $F_x =$ the fixed points of $G_x = \{y \in X : G_y \supseteq G_x\}$.
- $F'_x =$ the generic elements in $F_x = \{y \in X : G_y = G_x\}$.
- $N_x =$ the normaliser of G_x in $G = \{g \in G : gG_xg^{-1} = G_x\}$.
- $W_x =$ the Weyl group at $x \in X = N_x/G_x$.
- There is an action of N_x on G/G_x:

  $$n \bullet gG_x = gn^{-1}G_x,$$

  which induces a free action of W_x on G/G_x.
- Similarly, there is an action of N_x on F_x:

  $$n \bullet y = ny,$$

  which leaves F'_x invariant and induces a free action of W_x on F'_x. Consider the free diagonal action of W_x on $G/G_x \times F'_x$.

Fibration Theorem:

Theorem

1. *The map*

$$\phi : G/G_x \times F'_x \longrightarrow R(x), \qquad \phi(gG_x, y) = g \bullet y,$$

is well-defined.

2. *For each $z \in R(x)$, $\phi^{-1}(z)$ is an orbit of W_x.*
3. *The induced map*

$$\bar{\phi} : \{G/G_x \times F'_x\}/W_x \longrightarrow R(x)$$

is a bijection.

Ref: See [Kulkarni1]

The projection onto the first factor in the above bijection gives a "set-theoretic" bundle with fibers F'_x and base G/N_x.

The important special case: $X = G$, and the action of G on X is by conjugation.

In this case, $G_x =$ the centraliser of an element x in $X = G$,

and F_x = the center of the centraliser of x.

Note that F'_x := the generic elements of F_x, is a subset of an abelian group, and G/N_x is a homogeneous space of G.

Thus, via the bijection induced by $\bar{\phi}$, to an element x of $R(x)$ we can canonically associate an element of F_x and an element of G/N_x, which are its "numerical" and "spatial" invariants.

z-Classes and Isoclinism

P. Hall introduced an interesting notion of equivalence for all groups. Two groups G_1, G_2 are called <u>isoclinic</u> if

- Their commutator subgroups are isomorphic by an isomorphism ϕ.
- Their quotients by their respective centers are isomorphic by an isomorphism θ.
- The maps ϕ, θ are compatible with the natural map

$$G/Z(G) \times G/Z(G) \to G', \qquad (xZ(G), yZ(G)) \mapsto [x, y].$$

z-classes in finite groups

The notion of a z-class is closely related to isoclinism. In particular,

in two isoclinic groups, there is a canonical bijection between their z-classes. In particular, if a group has finitely many z-classes, then in its isoclinic family, each group has same number of z-classes.

We now wish to apply the philosophy of z-classes to the theory of finite groups. The finite groups with discrete topology, are, of course, compact Lie groups. But this is a different world! Some crucial results about compact Lie groups depend on the connected-ness hypothesis. For example, a compact connected solvable Lie group is abelian, and so it is a product of finitely many (possibly zero) copies of $S^{1'}s$.

On the other hand **finite solvable groups** is a major field of study by itself, with analogies with the field of non-compact solvable Lie groups!

Its subfield, namely **finite nilpotent groups**, is already a jungle! The only simplification is:

a finite nilpotent group is a direct product of its p-Sylow subgroups.

So the study of finite nilpotent groups essentially reduces to the study of p-groups.

At the other extreme, the classification of finite simple groups is a major achievement of mathematics in the 20th century. A major world-wide mathematical activity is to understand and simplify its proof, and develop a general theory of finite groups.

In this lecture I report only a few beginning results on z-classes in finite groups. It is a joint work with Rahul Kitture and Vikas Jadhav, and it is accepted for publication in the [Kulkarni-Kitture-Jadhav].

The results of interest here are

- the bounds on the number of z-classes
- connection with linear representations of groups.
- characterization of p-groups with minimum number of z-classes.

Results obtained:

1. In any group, a normal subgroup is a union of conjugacy classes. We obtain the following analogue:
   in any group, a maximal abelian normal subgroup[1] *is a union of z-classes.*
2. A non-abelian finite group contains at least 3 z-classes. Further, a finite group attaining this bound must be solvable but not nilpotent.
   This result is closely related to the **Wedderburn's Theorem** that *a finite division ring is a field.* cf. [Rony Gouraige].

[1] i.e. maximal among abelian normal subgroups

Connection with Representations over $\mathbb{Q}$

It is known that, in a finite group,

$$\#\begin{pmatrix}\text{conjugacy classes}\\ \text{of cyclic subgroups}\end{pmatrix} = \#\begin{pmatrix}\mathbb{Q}\text{-irreducible}\\ \text{representations}\end{pmatrix}$$

(cf. [Serre], §13.1, p. 103).

The proposition implies:

Theorem

Let G be a finite group, and $Z(G)$ its center. The number of z-classes is at most the number of irreducible representations of $G/Z(G)$ over $\mathbb{Q}$.

We have obtained more precise upper and lower bounds for *p*-groups, which are actually attained.

Upper Bound

Theorem

If $[G\colon Z(G)] = p^k$, the G has at most $\frac{p^k-1}{p-1} + 1$ z-classes.

Theorem (Necessary condition to attain bound)

If G is a p-group, which attains above upper bound on the number of z-classes then G is isoclinic to either a non-abelian group of order p^3 or a special p-group with no abelian maximal subgroup.

A p-group is said to be special if its center and commutator subgroup coincide and are elementary abelian p-groups.

The upper bound is attained, as shown by following extra-special p-group.

$$G = \left\{ \begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1,n+1} & a_{1,n+2} \\ & 1 & 0 & \cdots & 0 & a_{2,n+2} \\ & & \ddots & & & \vdots \\ & & & & 1 & a_{n+1,n+2} \\ & & & & & 1 \end{bmatrix} \right\}$$

Here $|G| = p^{2n+1}$, $Z(G)$ has index p^{2n} in G, and $G/Z(G)$ is an elementary abelian group of order p^{2n}. It has $\frac{p^{2n}-1}{p-1} + 1$ z-classes. Note that this number coincides with the number of conjugacy classes of cyclic subgroups in $G/Z(G)$, which is also the number of its irreducible representations over $\mathbb{Q}$.

Lower Bound

Theorem

Any non-abelian finite p-group contains at least $p + 2$ z-classes.

Theorem (Necessary and sufficient condition)

The lower bound is attained by a non-abelian finite p-group if and only if either $G/Z(G) \cong C_p \times C_p$ or the following holds:

1. *G has unique abelian subgroup of index p*
2. *the center of $G/Z(G)$ has order p.*

Theorem (Necessary and sufficient condition using isoclinism)

A p-group contains exactly $p + 2$ z-classes if and only if it is isoclinic to a p-group of maximal class with an abelian subgroup of index p.

Computation:

By the above necessary and sufficient conditions (to attain lower bound) it can be shown that

Theorem

Any non-abelian group of order p^3 or p^4 has exactly $p+2$ z-classes.

However, there are groups of order $\geq p^5$ with more than $p+2$ z-classes.

We have a more detailed information on z-classes in p-groups of order p^5, cf. [Jadhav-Kitture].

References:

1. Kulkarni R. S., *Dynamical types and conjugacy classes of centralizers in groups*, J. Ramanujan Math. Soc., **22**(2), 2007.
2. Kulkarni R. S., *Dynamics of Linear and Affine Maps*, Asian Journal of Mathematics, **12**(3), 2008.
3. K. Gongopadhyay, Kulkarni R. S., *z-Classes of isometries of the hyperbolic space*, Conformal Geometry and Dynamics **13**, 2009.
4. K. Gongopadhyay, Kulkarni R. S., *The z-classes of isometries*, J. Indian Math. Soc., **81**(3-4), 2014.
5. K. Gongopadhyay, *The z-classes of quaternionic hyperbolic isometries*, J. Group Theory, 16(**6**), 2013.
6. Rony Gouraige, *On the z-classes in centrally finite division rings*, Contemporary Mathematics vol. **639**, AMS, pp. 243-251.

7. R. S. Kulkarni, R. D. Kittue, V. Jadhav, *z-classes in groups*, Journal of Algebra and its Applications, [to appear].
8. V. Jadhav, R. D. Kitture, *z-classes in p-groups of order $\leq p^5$*, Bulletin of Allahabad Math. Society, 29(2), 2014, pp. 173-194.
9. J. P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, NY 1977.
10. A. K. Singh, *Conjugacy classes of centralizers in G_2*, J. Ramanujan Math. Soc. Vol. 23, No. 4 (2008), pp. 327-336.
11. Robert Steinberg, *Conjugacy Classes in Algebraic Groups*, notes by V. Deodhar, Lecture Notes in Mathematics **366**, Springer-Verlag, Berlin, 1974.

THANK YOU