Conjugation classes of pairs in $\mathrm{SL}(4, \mathbb{C})$ and $\mathrm{SU}(3,1)$

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(joint work with Sean Lawton)

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- This result is increamental in the development of Teichmüller theory. In particular, it gives the Fenchel-Nielsen coordinates on the Teichmüller space.

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- In geometric terms, answering this question will give some idea about the topology of the complex hyperbolic quasi-Fuchsian spaces in dimension three.

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- Let $V = \mathbb{C}^{n,1}$ be the complex vector space $\mathbb{C}^n$ equipped with the Hermitian form of signature (n,1) given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = -z_1 \bar{w}_1 + z_2 \bar{w}_2 + z_3 \bar{w}_3 + \cdots + +z_{n+1} \bar{w}_{n+1}.$$

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- We consider the following subspaces of $\mathbb{C}^{n,1}$:

$$V_- = \{\mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}, \ \mathbb{V}_+ = \{\mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0\},$$

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- The complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ is the projectivization of $\mathbb{V}_-$. It can be identified with the disk $\mathbb{D}^{2n}$. The ideal boundary is $\mathbb{S}^{2n-1}$.

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1. An isometry is *elliptic* if it fixes at least one point on $\mathbf{H}_{\mathbb{C}}^{n}$.
2. An isometry is *loxodromic* or *hyperbolic* if it fixes exactly two points of $\partial \mathbf{H}_{\mathbb{C}}^{n}$.
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Goldman classified these isometries algebraically in $\mathrm{SU}(2, 1)$. In a joint work with Parker and Parsad, we have generalized Goldman's result for $\mathrm{SU}(p, q)$. In particular, this gives a complete algebraic classification for $\mathrm{SU}(3, 1)$, the group of our interest in this talk!

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- The G-character variety of Γ is then the conjugation orbit space $\mathfrak{X}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)^*/G$.
- Our interest: $\Gamma = F_2$, $G = \mathrm{SL}(4, \mathbb{C})$ and $\mathrm{SU}(3, 1)$

Algebraic Topology of $\mathfrak{X}(\Gamma, G)$

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- **Theorem:** [Florentino-Lawton], [Luna]
  $\mathfrak{X}(\Gamma, G)$ is homeomorphic to the geometric points (with the Euclidean topology on an affine variety) of the Geometric Invariant Theory (GIT) quotient
  $\mathrm{Hom}(\Gamma, G) /\!\!/ G := \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$, where
  $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is the ring of G-invariant polynomials in the coordinate ring $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]$.

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- (Florentino-Lawton-Ramras) The GIT quotient with this topology is homotopic to the non-Hausdorff quotient space $\mathrm{Hom}(\Gamma, G)/G$.

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Let F_r^+ be the free non-commutative monoid generated by symbols $\{x_1, ..., x_r\}$. Let M_r^+ be the monoid generated by $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_r\}$, where $\mathbf{x}_k = (x_{ij}^k)$ are matrices in rn^2 indeterminates.

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There is a surjection $F_r^+ \to M_r^+$, defined by mapping $x_i \mapsto \mathbf{x}_i$. Let $\mathbf{w} \in M_r^+$ be the image of $w \in F_r^+$ under this map.

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(Procesi) The ring of invariants $\mathbb{C}[\mathfrak{gl}(n, \mathbb{C})^r]^{\mathrm{SL(n,\mathbb{C})}}$ is generated by

$$\{\mathrm{tr}(\mathbf{w}) \mid \mathrm{w} \in \mathrm{F_r^+},\ |\mathrm{w}| \leq \mathrm{n^2}\}. \tag{1}$$

The coordinate ring $\mathbb{C}[\mathfrak{X}(F_r, \mathrm{SL(n,\mathbb{C})})]$ is equal to $\mathbb{C}[\mathrm{SL(n,\mathbb{C})^r}]^{\mathrm{SL(n,\mathbb{C})}}$.

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where Δ is the ideal generated by the r polynomials $\det(\mathbf{x}_k) - 1$.

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Since the characteristic polynomial allows one to write the determinant as a polynomial in traces of words, $\mathbb{C}[\mathfrak{X}(F_r, \mathrm{SL(n,\mathbb{C})})]$ is generated by $\{\mathrm{tr}(\mathbf{w}) \mid \mathrm{w} \in \mathrm{F_r^+},\ |\mathrm{w}| \leq \mathrm{n}^2\}$ as well.

Some Notations

$\mathbf{x}_k^* = (-1)^{i+j}\mathrm{Cof}_{ji}(\mathbf{x}_k)$. Let M_r^* be the monoid generated by $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_r\}$ and $\{\mathbf{x}_1^*, \mathbf{x}_2^*, ..., \mathbf{x}_r^*\}$.

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Now let $\mathbb{C}M_r$ be the group algebra defined over $\mathbb{C}$ with respect to matrix addition and scalar multiplication in M_r. Likewise, let $\mathbb{C}M_r^*$ be the semi-group algebra of the monoid M_r^*.

A Non-commutative Diagram

The following diagram (from Lawton's thesis) forms a bridge between the non-commutative algebra $\mathbb{C}M_r$ and the moduli space $\mathfrak{X}(F_r, \mathrm{SL(n,\mathbb{C})})$, built in the language of free groups:

$$\begin{array}{ccccc} F_r^+ & \longrightarrow & F_r & & \\ \downarrow & & \downarrow & & \\ \mathbb{C}M_r^+ & \longrightarrow & \mathbb{C}M_r & \xrightarrow{\mathrm{tr}} & \mathbb{C}[\mathfrak{X}(F_r, \mathrm{SL(n,\mathbb{C})})]. \end{array}$$

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This relationship has been exploited to obtain many geometric results in the case $n = 3$. In particular, the relationship to the non-commutative algebra was used by Lawton to completely describe the coordinate ring of $\mathfrak{X}(F_2, \mathrm{SL}(3, \mathbb{C}))$.

$\mathfrak{X}(F_2, \mathrm{SL}(3,\mathbb{C}))$

(Lawton) $\mathfrak{X}(F_2, \mathrm{SL}(3,\mathbb{C}))$ is generated by:

$$\mathrm{tr}(\mathbf{x}),\ \mathrm{tr}(\mathbf{y}),\ \mathrm{tr}(\mathbf{x}\mathbf{y}^{-1}),\ \mathrm{tr}(\mathbf{x}^{-1}),\ \mathrm{tr}(\mathbf{y}^{-1}),$$

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As a consequence of this result, we have the following for $\mathrm{SU}(2,1)$.

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Parker and Platis proved a special case:

(Parker & Platis) *A pair of loxodromic elements in* $\mathrm{SU}(2,1)$ *is determined up to conjugacy by their traces and a point on the cross ratio variety corrresponding to these elements.*

Partial Generalization of Parker-Platis for $\mathrm{SU}(3,1)$

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(ii) The fixed point set of A is disjoint from at least one of the $\mathbb{C}^2$-chains polar to the positive eigenvectors of B and, the fixed point set of B is disjoint from at least one of the $\mathbb{C}^2$-chains polar to the positive eigenvectors of A.

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Theorem. (G. - Parsad) *Let $\rho : F_2 \to \mathrm{SU}(3,1)$ be a representation such that $\rho(m)$, $\rho(n)$ are loxodromic and generate a non-singular subgroup of $\mathrm{SU}(3,1)$. Then for some $i, j \in \{1,2\}$, there exists two non-zero complex parameters α_i and β_j such that these, along with coefficients of the characteristic polynomials of $\rho(m)$, $\rho(n)$ and a point on the cross-ratio variety, completely determine ρ up to conjugacy.*

Here α_1, α_2, β_1, β_2 are 'cross-ratios' with three null vectors and one positive-type eigenvector.

Generating Set for $\mathfrak{X}(F_2, \mathrm{SL}(4,\mathbb{C}))$

Word Length	Generator
1	$\mathrm{tr}(\mathbf{x}),\ \mathrm{tr}(\mathbf{y})$
2	$\mathrm{tr}(\mathbf{x}^2),\ \mathrm{tr}(\mathbf{xy}),\ \mathrm{tr}(\mathbf{y}^2)$
3	$\mathrm{tr}(\mathbf{x}^3),\ \mathrm{tr}(\mathbf{x}^2\mathbf{y}),\ \mathrm{tr}(\mathbf{xy}^2),\ \mathrm{tr}(\mathbf{y}^3)$
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Word Length	Generator
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7	$\mathrm{tr}(\mathbf{x}^3\mathbf{y}^2\mathbf{xy}),\ \mathrm{tr}(\mathbf{y}^3\mathbf{x}^2\mathbf{yx})$
8	$\mathrm{tr}(\mathbf{x}^3\mathbf{y}^2\mathbf{x}^2\mathbf{y}),\ \mathrm{tr}(\mathbf{y}^3\mathbf{x}^2\mathbf{y}^2\mathbf{x}),\ \mathrm{tr}(\mathbf{x}^3\mathbf{y}^3\mathbf{xy}),\ \mathrm{tr}(\mathbf{y}^3\mathbf{x}^3\mathbf{yx})$
9	$\mathrm{tr}(\mathbf{x}^3\mathbf{yx}^2\mathbf{yxy}),\ \mathrm{tr}(\mathbf{x}^2\mathbf{y}^2\mathbf{xyx}^2\mathbf{y}),$ $\mathrm{tr}(\mathbf{y}^2\mathbf{x}^2\mathbf{yxy}^2\mathbf{x}),\ \mathrm{tr}(\mathbf{y}^3\mathbf{xy}^2\mathbf{xyx})$
10	$\mathrm{tr}(\mathbf{x}^3\mathbf{y}^3\mathbf{x}^2\mathbf{y}^2)$

Table: $\mathcal{G}_2$

Djoković, and independently, Drensky and Sadikova showed that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a minimal system of 32 generators for $\mathbb{C}[\mathfrak{gl}(4,\mathbb{C})^2 /\!\!/ \mathrm{SL}(4,\mathbb{C})]$, where $\mathcal{G}_1$ is a system of parameters (in particular, a maximal algebraically independent set). We could improve this number by two for $\mathbb{C}[\mathfrak{X}(F_2, \mathrm{SL}(4,\mathbb{C}))]$.

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Theorem

(G. - Lawton) $\mathcal{G}_1 \cup \mathcal{G}_2 - \{\mathrm{tr}(\mathbf{x}^4),\ \mathrm{tr}(\mathbf{y}^4)\}$ *is a minimal system of* 30 *generators for* $\mathbb{C}[\mathfrak{X}(F_2, \mathrm{SL}(4,\mathbb{C}))]$, *where* $\mathcal{G}_1 - \{\mathrm{tr}(\mathbf{x}^4),\ \mathrm{tr}(\mathbf{y}^4)\}$ *is a maximal set of* 15 *algebraically independent elements.*

Lemma

Let $\mathcal{G}$ *generate* $\mathbb{C}[\mathfrak{X}(F_r, \mathrm{SL}(4,\mathbb{C}))]$ *and suppose* $\mathrm{tr}(\mathbf{ux}^3\mathbf{v})$ *is in* $\mathcal{G}$*. Then* $\mathcal{G} \cup \{\mathrm{tr}(\mathbf{ux}^{-1}\mathbf{v})\} - \{\mathrm{tr}(\mathbf{ux}^3\mathbf{v})\}$ *remains a generating set as long as* $\mathrm{tr}(\mathbf{x})$, $\mathrm{tr}(\mathbf{x}^2)$, $\mathrm{tr}(\mathbf{x}^{-1})$, $\mathrm{tr}(\mathbf{uv})$, $\mathrm{tr}(\mathbf{uxv})$, *and* $\mathrm{tr}(\mathbf{ux}^2\mathbf{v})$ *are in the subring generated by* $\mathcal{G} - \{\mathrm{tr}(\mathbf{ux}^3\mathbf{v})\}$.

Proof.

The characteristic polynomial for $\mathrm{SL}(4,\mathbb{C})$ is:

$$\mathbf{x}^4 - \mathrm{tr}(\mathbf{x})\mathbf{x}^3 + \left(\frac{\mathrm{tr}(\mathbf{x})^2 - \mathrm{tr}(\mathbf{x}^2)}{2}\right)\mathbf{x}^2 - \mathrm{tr}(\mathbf{x}^{-1})\mathbf{x} + \mathbf{I} = 0.$$

Multiplying through on the left by a word $\mathbf{u}$ and on the right by $\mathbf{x}^{-1}\mathbf{v}$ for a word $\mathbf{v}$ gives:

$$\mathbf{ux}^3\mathbf{v} - \mathrm{tr}(\mathbf{x})\mathbf{ux}^2\mathbf{v} + \left(\frac{\mathrm{tr}(\mathbf{x})^2 - \mathrm{tr}(\mathbf{x}^2)}{2}\right)\mathbf{uxv} - \mathrm{tr}(\mathbf{x}^{-1})\mathbf{uv} + \mathbf{ux}^{-1}\mathbf{v} = 0.$$

Therefore, taking traces of both sides of this latter equation, we have the lemma. □

Let τ be the involutive outer automorphism that permutes $\mathbf{x}$ and $\mathbf{y}$ and let ι be the involutive outer automorphism that sends $\mathbf{x} \mapsto \mathbf{x}^{-1}$ and $\mathbf{y} \mapsto \mathbf{y}^{-1}$. Clearly τ and ι act on $\mathfrak{X}(F_2, \mathrm{SL}(4, \mathbb{C}))$ and its coordinate ring.

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Corollary

$\mathcal{S} \cup \tau(\mathcal{S}) \cup \{\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{x}^2\mathbf{y}^2)\}$ *is a minimal set of 30 generators for* $\mathbb{C}[\mathfrak{X}(F_2, \mathrm{SL}(4,\mathbb{C}))]$, *where,* Let τ be the involutive outer automorphism that permutes $\mathbf{x}$ and $\mathbf{y}$ and let ι be the involutive outer automorphism that sends $\mathbf{x} \mapsto \mathbf{x}^{-1}$ and $\mathbf{y} \mapsto \mathbf{y}^{-1}$. Clearly τ and ι act on $\mathfrak{X}(F_2, \mathrm{SL}(4, \mathbb{C}))$ and its coordinate ring.

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Word Length	Generator
1	$\mathrm{tr}(\mathbf{x})$
2	$\mathrm{tr}(\mathbf{x}^2),\ \mathrm{tr}(\mathbf{xy})$
3	$\mathrm{tr}(\mathbf{x}^{-1}),\ \mathrm{tr}(\mathbf{xy}^{-2})$
4	$\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}),\ \mathrm{tr}(\mathbf{x}^2\mathbf{y}^2),\ \mathrm{tr}(\mathbf{xyxy})$
5	$\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^2)$
6	$\mathrm{tr}((\mathbf{x}^2\mathbf{y})^2), \mathrm{tr}(\mathbf{x}^2\mathbf{y}^2\mathbf{xy})$
7	$\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^2\mathbf{xy})$
8	$\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^2\mathbf{x}^2\mathbf{y}),\ \mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{xy})$
9	$\mathrm{tr}(\mathbf{x}^{-1}\mathbf{yx}^2\mathbf{yxy}),\ \mathrm{tr}(\mathbf{x}^2\mathbf{y}^2\mathbf{xyx}^2\mathbf{y})$

Table: $\mathcal{S}$

Theorem

(G. - Lawton) *The following 22 traces determine any (polystable) pair* $\langle A, B\rangle$ *up to conjugation where* $A, B \in \mathrm{SU}(3,1)$*:*

Word Length	*Generator*
1	$\mathrm{tr}(\mathbf{x})$, $\mathrm{tr}(\mathbf{y})$
2	$\mathrm{tr}(\mathbf{x}^2)$, $\mathrm{tr}(\mathbf{xy})$, $\mathrm{tr}(\mathbf{y}^2)$, $\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y})$
3	$\mathrm{tr}(\mathbf{xy}^2)$, $\mathrm{tr}(\mathbf{yx}^2)$
4	$\mathrm{tr}(\mathbf{x}^2\mathbf{y}^2)$, $\mathrm{tr}(\mathbf{xyxy})$, $\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{xy})$
5	$\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^2\mathbf{xy})$, $\mathrm{tr}(\mathbf{y}^{-1}\mathbf{x}^2\mathbf{yx})$
6	$\mathrm{tr}((\mathbf{x}^2\mathbf{y})^2)$, $\mathrm{tr}((\mathbf{y}^2\mathbf{x})^2)$, $\mathrm{tr}(\mathbf{x}^2\mathbf{y}^2\mathbf{xy})$, $\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{x}^2\mathbf{y}^2)$ $\mathrm{tr}(\mathbf{y}^2\mathbf{x}^2\mathbf{yx})$, $\mathrm{tr}(\mathbf{x}^{-1}\mathbf{y}^2\mathbf{x}^2\mathbf{y})$, $\mathrm{tr}(\mathbf{y}^{-1}\mathbf{x}^2\mathbf{y}^2\mathbf{x})$
7	$\mathrm{tr}(\mathbf{x}^{-1}\mathbf{yx}^2\mathbf{yxy})$, $\mathrm{tr}(\mathbf{y}^{-1}\mathbf{xy}^2\mathbf{xyx})$

Thank You!