Conjugation classes of pairs in $SL(4, \mathbb{C})$ and SU(3, 1)

Krishnendu Gongopadhyay

(joint work with Sean Lawton)

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Theorem of Fricke and Vogt

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Actually it holds more generally for any 'polystable' pairs.

• This result is increamental in the development of Teichmüller theory. In particular, it gives the Fenchel-Nielsen coordinates on the Teichmüller space.

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- We shall address this question for ${\rm SL}(4,\mathbb{C})$ and ${\rm SU}(3,1)$ in this talk.
- In geometric terms, answering this question will give some idea about the topology of the complex hyperbolic quasi-Fuchsian spaces in dimension three.

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 Let V = C^{n,1} be the complex vector space Cⁿ equipped with the Hermitian form of signature (n,1) given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = -z_1 \bar{w}_1 + z_2 \bar{w}_2 + z_3 \bar{w}_3 + \dots + z_{n+1} \bar{w}_{n+1}$$

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• We consider the following subspaces of $\mathbb{C}^{n,1}$:

$$\begin{split} V_{-} &= \{ \mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0 \}, \ \mathbb{V}_{+} = \{ \mathbf{z} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0 \}, \\ V_{0} &= \{ \mathbf{z} - \{ \mathbf{0} \} \in \mathbb{C}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle = 0 \}. \end{split}$$

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 The complex hyperbolic space Hⁿ_ℂ is the projectivization of *V*_−. It can be identified with the disk D²ⁿ. The ideal boundary is S^{2n−1}.
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- **(**) An isometry is *elliptic* if it fixes at least one point on $\mathbf{H}^n_{\mathbb{C}}$.
- ② An isometry is *loxodromic* or *hyperbolic* if it fixes exactly two points of ∂Hⁿ_C.
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Goldman classified these isometries algebraically in SU(2, 1). In a joint work with Parker and Parsad, we have generalized Goldman's result for SU(p, q). In particular, this gives a complete algebraic classification for SU(3, 1), the group of our interest in this talk!

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- Our interest: $\Gamma = F_2$, $G = SL(4, \mathbb{C})$ and SU(3, 1)

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- Theorem: [Florentino-Lawton], [Luna] *X*(Γ, G) is homeomorphic to the geometric points (with the Euclidean topology on an affine variety) of the Geometric Invariant Theory (GIT) quotient Hom(Γ, G)//G := Spec(ℂ[Hom(Γ, G)]^G), where ℂ[Hom(Γ, G)]^G is the ring of G-invariant polynomials in the coordinate ring ℂ[Hom(Γ, G)].

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- (Florentino-Lawton-Ramras) The GIT quotient with this topology is homotopic to the non-Hausdorff quotient space $\operatorname{Hom}(\Gamma, G)/G$.

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Let F_r^+ be the free non-commutative monoid generated by symbols $\{x_1, ..., x_r\}$. Let M_r^+ be the monoid generated by $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_r\}$, where $\mathbf{x}_k = (x_{ii}^k)$ are matrices in rn^2 indeterminates.

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There is a surjection $F_r^+ \to M_r^+$, defined by mapping $x_i \mapsto \mathbf{x}_i$. Let $\mathbf{w} \in M_r^+$ be the image of $w \in F_r^+$ under this map.

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(Procesi) The ring of invariants $\mathbb{C}[\mathfrak{gl}(n,\mathbb{C})^r]^{\mathrm{SL}(n,\mathbb{C})}$ is generated by

$$\{\mathrm{tr}(\mathbf{w})\mid \mathrm{w}\in\mathrm{F}_{\mathrm{r}}^{+},\ |\mathrm{w}|\leq\mathrm{n}^{2}\}. \tag{1}$$

The coordinate ring $\mathbb{C}[\mathfrak{X}(F_r, SL(n, \mathbb{C}))]$ is equal to $\mathbb{C}[SL(n, \mathbb{C})^r]^{SL(n, \mathbb{C})}$.

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$$(\mathbb{C}[\mathfrak{gl}(n,\mathbb{C})^r]/\Delta)^{\mathrm{SL}(n,\mathbb{C})} = \mathbb{C}[\mathfrak{gl}(n,\mathbb{C})^r]^{\mathrm{SL}(n,\mathbb{C})}/\Delta,$$

where Δ is the ideal generated by the *r* polynomials det(\mathbf{x}_k) - 1.

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Since the characteristic polynomial allows one to write the determinant as a polynomial in traces of words, $\mathbb{C}[\mathfrak{X}(F_r, \mathrm{SL}(n, \mathbb{C}))]$ is generated by $\{\mathrm{tr}(\mathbf{w}) \mid w \in F_r^+, |w| \le n^2\}$ as well.

 $\begin{aligned} \mathbf{x}_k^* &= (-1)^{i+j} \mathrm{Cof}_{ji}(\mathbf{x}_k). \text{ Let } M_r^* \text{ be the monoid generated by} \\ \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_r\} \text{ and } \{\mathbf{x}_1^*, \mathbf{x}_2^*, ..., \mathbf{x}_r^*\}. \end{aligned}$

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Let N_r be the normal sub-monoid generated by $\{\det(\mathbf{x}_k)\mathbf{I} \mid 1 \le k \le r\}.$

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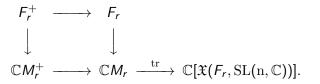
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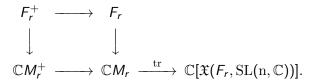
Now let $\mathbb{C}M_r$ be the group algebra defined over \mathbb{C} with respect to matrix addition and scalar multiplication in M_r . Likewise, let $\mathbb{C}M_r^*$ be the semi-group algebra of the monoid M_r^* .

The following diagram (from Lawton's thesis) forms a bridge between the non-commutative algebra $\mathbb{C}M_r$ and the moduli space $\mathfrak{X}(F_r, \mathrm{SL}(n, \mathbb{C}))$, built in the language of free groups:



This relationship has been exploited to obtain many geometric results in the case n = 3.

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This relationship has been exploited to obtain many geometric results in the case n = 3. In particular, the relationship to the non-commutative algebra was used by Lawton to completely describe the coordinate ring of $\mathfrak{X}(F_2, \mathrm{SL}(3, \mathbb{C}))$.

$\mathfrak{X}(F_2,\mathrm{SL}(3,\mathbb{C}))$

(Lawton) $\mathfrak{X}(F_2, \mathrm{SL}(3, \mathbb{C}))$ is generated by: $\operatorname{tr}(\mathbf{x}), \operatorname{tr}(\mathbf{y}), \operatorname{tr}(\mathbf{x}\mathbf{y}^{-1}), \operatorname{tr}(\mathbf{x}^{-1}), \operatorname{tr}(\mathbf{y}^{-1}),$ $\operatorname{tr}(\mathbf{x}^{-1}\mathbf{y}^{-1}), \operatorname{tr}(\mathbf{x}^{-1}\mathbf{y}), \operatorname{tr}([\mathbf{x}, \mathbf{y}]),$ where $[\mathbf{x}, \mathbf{y}] = \mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{x}\mathbf{y}.$

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where $[\mathbf{x}, \mathbf{y}] = \mathbf{x}^{-1} \mathbf{y}^{-1} \mathbf{x} \mathbf{y}$.

As a consequence of this result, we have the following for SU(2, 1). (Will, Wen) $\mathfrak{X}(F_2, SU(2, 1))$ is generated by:

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Parker and Platis proved a special case:

(Parker & Platis) A pair of loxodromic elements in SU(2,1) is determined up to conjugacy by their traces and a point on the cross ratio variety corrresponding to these elements.

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Theorem. (G. - Parsad) Let $\rho : F_2 \to SU(3,1)$ be a representation such that $\rho(m)$, $\rho(n)$ are loxodromic and generate a non-singular subgroup of SU(3,1). Then for some $i, j \in \{1,2\}$, there exists two non-zero complex parameters α_i and β_j such that these, along with coefficients of the characteristic polynomials of $\rho(m)$, $\rho(n)$ and a point on the cross-ratio variety, completely determine ρ up to conjugacy.

Here α_1 , α_2 , β_1 , β_2 are 'cross-ratios' with three null vectors and one positive-type eigenvector.

Generating Set for $\mathfrak{X}(F_2, \mathrm{SL}(4, \mathbb{C}))$

Word Length	Generator
1	$\operatorname{tr}(\mathbf{x}), \ \operatorname{tr}(\mathbf{y})$
2	$\operatorname{tr}(\mathbf{x}^2), \ \operatorname{tr}(\mathbf{x}\mathbf{y}), \ \operatorname{tr}(\mathbf{y}^2)$
3	$\operatorname{tr}(\mathbf{x}^3), \ \operatorname{tr}(\mathbf{x}^2\mathbf{y}), \ \operatorname{tr}(\mathbf{x}\mathbf{y}^2), \ \operatorname{tr}(\mathbf{y}^3)$
4	$\operatorname{tr}(\mathbf{x}^4), \operatorname{tr}(\mathbf{x}^3\mathbf{y}), \operatorname{tr}(\mathbf{x}^2\mathbf{y}^2), \operatorname{tr}(\mathbf{x}\mathbf{y}^3), \operatorname{tr}(\mathbf{y}^4), \operatorname{tr}(\mathbf{x}\mathbf{y}\mathbf{x}\mathbf{y})$
6	$\operatorname{tr}((\mathbf{x}^2\mathbf{y})^2), \ \operatorname{tr}((\mathbf{y}^2\mathbf{x})^2)$

Table: \mathcal{G}_1

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Word Length	Generator
5	$\operatorname{tr}(\mathbf{x}^3\mathbf{y}^2), \ \operatorname{tr}(\mathbf{y}^3\mathbf{x}^2)$
6	$tr(x^2y^2xy), tr(y^2x^2yx)$
7	$tr(x^3y^2xy), tr(y^3x^2yx)$
8	$tr(\mathbf{x}^3\mathbf{y}^2\mathbf{x}^2\mathbf{y}), tr(\mathbf{y}^3\mathbf{x}^2\mathbf{y}^2\mathbf{x}), tr(\mathbf{x}^3\mathbf{y}^3\mathbf{x}\mathbf{y}), tr(\mathbf{y}^3\mathbf{x}^3\mathbf{y}\mathbf{x})$
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10	$\operatorname{tr}(\mathbf{x}^3\mathbf{y}^3\mathbf{x}^2\mathbf{y}^2)$

Table: \mathcal{G}_2

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Krishnendu Gongopadhyay (joint work with Sean Lawton) Conjugation classes of pairs

Djoković, and independently, Drensky and Sadikova showed that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a minimal system of 32 generators for $\mathbb{C}[\mathfrak{gl}(4,\mathbb{C})^2/\!/\mathrm{SL}(4,\mathbb{C})]$, where \mathcal{G}_1 is a system of parameters (in particular, a maximal algebraically independent set). We could improve this number by two for $\mathbb{C}[\mathfrak{X}(F_2,\mathrm{SL}(4,\mathbb{C}))]$.

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Theorem

(G. - Lawton) $\mathcal{G}_1 \cup \mathcal{G}_2 - \{\operatorname{tr}(\mathbf{x}^4), \operatorname{tr}(\mathbf{y}^4)\}\$ is a minimal system of 30 generators for $\mathbb{C}[\mathfrak{X}(F_2, \operatorname{SL}(4, \mathbb{C}))]$, where $\mathcal{G}_1 - \{\operatorname{tr}(\mathbf{x}^4), \operatorname{tr}(\mathbf{y}^4)\}\$ is a maximal set of 15 algebraically independent elements.

Lemma

Let \mathcal{G} generate $\mathbb{C}[\mathfrak{X}(F_r, \mathrm{SL}(4, \mathbb{C}))]$ and suppose $\mathrm{tr}(\mathbf{ux}^3\mathbf{v})$ is in \mathcal{G} . Then $\mathcal{G} \cup \{\mathrm{tr}(\mathbf{ux}^{-1}\mathbf{v})\} - \{\mathrm{tr}(\mathbf{ux}^3\mathbf{v})\}$ remains a generating set as long as $\mathrm{tr}(\mathbf{x})$, $\mathrm{tr}(\mathbf{x}^2)$, $\mathrm{tr}(\mathbf{x}^{-1})$, $\mathrm{tr}(\mathbf{uv})$, $\mathrm{tr}(\mathbf{ux}\mathbf{v})$, and $\mathrm{tr}(\mathbf{ux}^2\mathbf{v})$ are in the subring generated by $\mathcal{G} - \{\mathrm{tr}(\mathbf{ux}^3\mathbf{v})\}$.

Proof.

The characteristic polynomial for $SL(4, \mathbb{C})$ is:

$$\mathbf{x}^{4} - \operatorname{tr}(\mathbf{x})\mathbf{x}^{3} + \left(\frac{\operatorname{tr}(\mathbf{x})^{2} - \operatorname{tr}(\mathbf{x}^{2})}{2}\right)\mathbf{x}^{2} - \operatorname{tr}(\mathbf{x}^{-1})\mathbf{x} + \mathbf{I} = 0.$$

Multiplying through on the left by a word ${\bm u}$ and on the right by ${\bm x}^{-1} {\bm v}$ for a word ${\bm v}$ gives:

$$\mathbf{u}\mathbf{x}^{3}\mathbf{v} - \mathrm{tr}(\mathbf{x})\mathbf{u}\mathbf{x}^{2}\mathbf{v} + \left(\frac{\mathrm{tr}(\mathbf{x})^{2} - \mathrm{tr}(\mathbf{x}^{2})}{2}\right)\mathbf{u}\mathbf{x}\mathbf{v} - \mathrm{tr}(\mathbf{x}^{-1})\mathbf{u}\mathbf{v} + \mathbf{u}\mathbf{x}^{-1}\mathbf{v} = 0.$$

Therefore, taking traces of both sides of this latter equation, we have the lemma. $\hfill \Box$

Let τ be the involutive outer automorphism that permutes **x** and **y** and let ι be the involutive outer automorphism that sends $\mathbf{x} \mapsto \mathbf{x}^{-1}$ and $\mathbf{y} \mapsto \mathbf{y}^{-1}$. Clearly τ and ι act on $\mathfrak{X}(F_2, \mathrm{SL}(4, \mathbb{C}))$ and its coordinate ring. Let τ be the involutive outer automorphism that permutes **x** and **y** and let ι be the involutive outer automorphism that sends $\mathbf{x} \mapsto \mathbf{x}^{-1}$ and $\mathbf{y} \mapsto \mathbf{y}^{-1}$. Clearly τ and ι act on $\mathfrak{X}(F_2, \mathrm{SL}(4, \mathbb{C}))$ and its coordinate ring.

Corollary

 $S \cup \tau(S) \cup \{ tr(\mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{x}^2\mathbf{y}^2) \}$ is a minimal set of 30 generators for $\mathbb{C}[\mathfrak{X}(F_2, SL(4, \mathbb{C}))]$, where,

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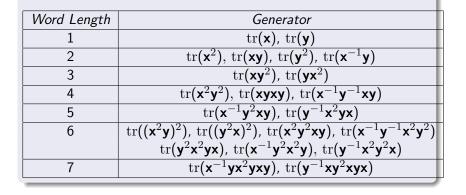
Word Length	Generator
1	$\operatorname{tr}(\mathbf{x})$
2	$tr(x^2), tr(xy)$
3	$\operatorname{tr}(\mathbf{x}^{-1}), \ \operatorname{tr}(\mathbf{x}\mathbf{y}^{-2})$
4	$\operatorname{tr}(\mathbf{x}^{-1}\mathbf{y}), \ \operatorname{tr}(\mathbf{x}^2\mathbf{y}^2), \ \operatorname{tr}(\mathbf{x}\mathbf{y}\mathbf{x}\mathbf{y})$
5	$\operatorname{tr}(\mathbf{x}^{-1}\mathbf{y}^2)$
6	$\operatorname{tr}((\mathbf{x}^2\mathbf{y})^2), \operatorname{tr}(\mathbf{x}^2\mathbf{y}^2\mathbf{x}\mathbf{y})$
7	$\operatorname{tr}(\mathbf{x}^{-1}\mathbf{y}^2\mathbf{x}\mathbf{y})$
8	$\operatorname{tr}(\mathbf{x}^{-1}\mathbf{y}^2\mathbf{x}^2\mathbf{y}), \ \operatorname{tr}(\mathbf{x}^{-1}\mathbf{y}^{-1}\mathbf{x}\mathbf{y})$
9	$\operatorname{tr}(\mathbf{x}^{-1}\mathbf{y}\mathbf{x}^{2}\mathbf{y}\mathbf{x}\mathbf{y}), \ \operatorname{tr}(\mathbf{x}^{2}\mathbf{y}^{2}\mathbf{x}\mathbf{y}\mathbf{x}^{2}\mathbf{y})$

Table: \mathcal{S}

Generating set for $\mathbb{C}[\mathfrak{X}(F_2, \mathrm{SU}(3, 1))]$

Theorem

(G. - Lawton) The following 22 traces determine any (polystable) pair $\langle A, B \rangle$ up to conjugation where $A, B \in SU(3, 1)$:



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Thank You!

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