# Minimal Surfaces and Complex Analysis

## Lecture 1: Survey

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# Lecture 1: Survey

I will lecture on the connection between Minimal Surfaces in Euclidean Spaces and Complex Analysis.

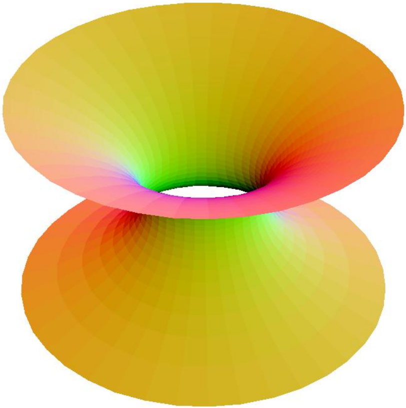
- A brief history of minimal surfaces.
- Basics on minimal surfaces and holomorphic null curves.
- Isotopies of conformal minimal immersions.
- Desingularizing conformal minimal immersions in $\mathbb{R}^n$ $(n \geq 5)$.
- Proper conformal minimal immersions to $\mathbb{R}^3$ and embeddings to $\mathbb{R}^5$
- On the Calabi-Yau problem for minimal surfaces.

Based on joint work with

- **Barbara Drinovec Drnovšek** and **Franc Forstnerič**, University of Ljubljana.
- **Francisco J. López**, University of Granada.

# Minimal Surfaces in $\mathbb{R}^3$

1744 **Euler** The only **area minimizing surfaces** of rotation in $\mathbb{R}^3$ are planes and catenoids.



# Minimal Surfaces in $\mathbb{R}^3$

1760 **Lagrange** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $f : \overline{\Omega} \to \mathbb{R}$ be a smooth function. Then the graph

$$S = \{(x, y, f(x, y)) \colon (x, y) \in \overline{\Omega}\} \subset \mathbb{R}^3$$

is an **area minimizing surface** if and only if

$$(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0;$$

equivalently,

$$\mathrm{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) = 0. \tag{1}$$

This is known as the **equation of minimal graphs**.

# Minimal Surfaces in $\mathbb{R}^3$

1776 **Meusnier** A (smooth) surface $S \subset \mathbb{R}^3$ satisfies locally the above equation iff its mean curvature function vanishes identically.

## Definition (Minimal Surface)

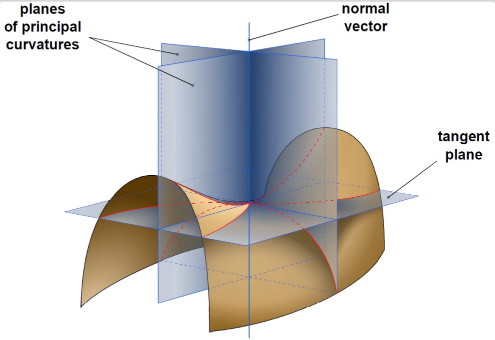
A smoothly immersed surface $M \to \mathbb{R}^3$ is said to be a **minimal surface** if its **mean curvature function** $\mathrm{H} \colon M \to \mathbb{R}$ is identically zero: $\mathrm{H} = 0$.

$$\mathrm{H} = \frac{\kappa_1 + \kappa_2}{2}$$

$$\mathrm{K} = \kappa_1 \kappa_2$$

where $\kappa_1$, $\kappa_2$ are the **principal curvatures**.

$\mathrm{H} = 0 \Rightarrow K = \kappa_1 \kappa_2 \leq 0$ (**Gauss curvature**).



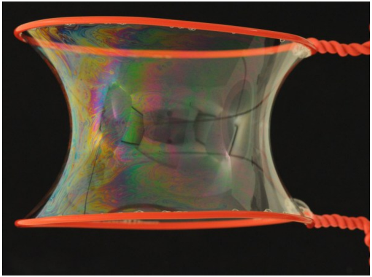
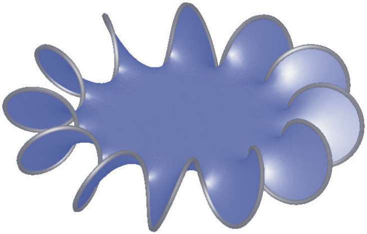
# Minimal Surfaces in $\mathbb{R}^3$

1776 **Meusnier** The helicoid is a minimal surface.

1842 **Catalan** The helicoid and the plane are the only ruled minimal surfaces in $\mathbb{R}^3$ (unions of straight lines).

# Local Theory: Plateau Problem

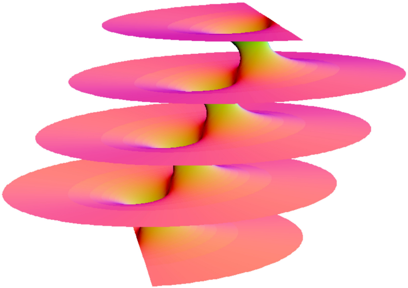
1873 **Plateau** Minimal surfaces can be physically obtained as soap films.

1932 **Douglas, Radó** Every continuous injective closed curve in $\mathbb{R}^n$ $(n \geq 3)$ spans a minimal surface.

# Examples by Riemann

1865 On the way to this solution, **Riemann** and others discovered new examples of minimal surfaces using the **Weierstrass Representation**.



2015 **Meeks, Pérez, Ros** Riemann's minimal examples, catenoids, helicoids, and planes are the only properly embedded minimal planar domains in $\mathbb{R}^3$.

# Minimal graphs minimize area

## Proposition

*Let $D$ be a smoothly bounded, compact domain in $\mathbb{R}^2_{(x,y)}$, let $f \colon D \to \mathbb{R}$ be a smooth function, and let $S := \mathrm{Graph}(f)$ be the graph of $f$.*

*If $S$ is minimal and $\widetilde{S} := \mathrm{Graph}(\widetilde{f})$, where $\widetilde{f} \colon D \to \mathbb{R}$ is a smooth function such that $\widetilde{f}|_{bD} = f|_{bD}$ (equivalently, $b\widetilde{S} = bS$), then the areas*

$$\mathscr{A}(S) \leq \mathscr{A}(\widetilde{S}).$$

*Furthermore, $\mathscr{A}(\widetilde{S}) = \mathscr{A}(S)$ if and only if $\widetilde{f} = f$.*

The same holds for minimal graphs in $\mathbb{R}^n$ for all $n \geq 3$. A graph $\{(u, f(u)) \colon u \in \overline{\Omega}\} \subset \mathbb{R}^n$ $(n \geq 3)$ over a smoothly bounded compact domain $\overline{\Omega} \subset \mathbb{R}^{n-1}$ minimizes volume if and only if satisfies the minimal surface equation (1).

# Minimal graphs minimize area. Proof

Let $\mathrm{N}\colon D \to \mathbb{S}^2$ be the normal map of the minimal surface

$$S = \{(x, y, f(x, y))\colon (x, y) \in D\}.$$

Consider the unitary vector field $V\colon D \times \mathbb{R} \to \mathbb{R}^3$ given by

$$V(x, y, z) = \mathrm{N}(x, y) = \frac{1}{\sqrt{1 + |\nabla f|^2}}(-f_x, -f_y, 1).$$

Computing,

$$\begin{aligned} \mathrm{div}_{\mathbb{R}^3} V &= \frac{\partial}{\partial x}\left(-\frac{f_x}{\sqrt{1+|\nabla f|^2}}\right) + \frac{\partial}{\partial y}\left(-\frac{f_y}{\sqrt{1+|\nabla f|^2}}\right) \\ &= -\mathrm{div}_{\mathbb{R}^2}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0. \end{aligned}$$

# Minimal graphs minimize area. Proof

Assume $S, \widetilde{S} \subset \{z > 0\}$. Let $W$ and $\widetilde{W}$ be the regions in $D \times \mathbb{R}$ with boundaries $bW = S \cup C \cup D$ and $b\widetilde{W} = \widetilde{S} \cup C \cup D$, where $C \subset bD \times \mathbb{R}$. By the divergence theorem,

$$\begin{aligned} 0 &= \int_W \operatorname{div}(V)\, dxdydz = \int_{bW} \langle V, \nu_{bW} \rangle d\mathscr{A}_{bW} \\ &= \int_D \langle V, \nu_D \rangle dxdy + \int_C \langle V, \nu_C \rangle d\mathscr{A}_C + \int_S \langle V, \nu_S \rangle d\mathscr{A}_S \end{aligned}$$

($\nu_\bullet$ =outer normal, $d\mathscr{A}_\bullet$=area element)

$$\begin{aligned} 0 &= \int_{\widetilde{W}} \operatorname{div}(V)\, dxdydz = \int_{b\widetilde{W}} \langle V, \nu_{b\widetilde{W}} \rangle d\mathscr{A}_{b\widetilde{W}} \\ &= \int_D \langle V, \nu_D \rangle dxdy + \int_C \langle V, \nu_C \rangle d\mathscr{A}_C + \int_{\widetilde{S}} \langle V, \nu_{\widetilde{S}} \rangle d\mathscr{A}_{\widetilde{S}}. \end{aligned}$$

Hence,

$$\int_S \langle V, \nu_S \rangle d\mathscr{A}_S = \int_{\widetilde{S}} \langle V, \nu_{\widetilde{S}} \rangle d\mathscr{A}_{\widetilde{S}}. \tag{2}$$

# Minimal graphs minimize area. Proof

We have $V = \mathrm{N} = \nu_S$ on $S$, and so

$$\int_S \langle V, \nu_S \rangle d\mathscr{A}_S = \int_S d\mathscr{A}_S = \mathscr{A}(S).$$

On the other hand, by Schwarz, $|\langle V, \nu_{\widetilde{S}} \rangle| \leq \|V\| \cdot \|\nu_{\widetilde{S}}\| = 1$, hence

$$\int_{\widetilde{S}} \langle V, \nu_{\widetilde{S}} \rangle d\mathscr{A}_{\widetilde{S}} \leq \int_{\widetilde{S}} d\mathscr{A}_{\widetilde{S}} = \mathscr{A}(\widetilde{S}).$$

Therefore, $\mathscr{A}(S) \leq \mathscr{A}(\widetilde{S})$ in view of (2).

If $\mathscr{A}(\widetilde{S}) = \mathscr{A}(S)$, then $|\langle V, \nu_{\widetilde{S}} \rangle| = \|V\| \cdot \|\nu_{\widetilde{S}}\| = \|V\| \cdot \|\nu_S\|$, hence $\nu_{\widetilde{S}} = \nu_S$ and so $\nabla \widetilde{f} = \nabla f$ and $\widetilde{f} - f$ is constant.
Since $\widetilde{f}|_{bD} = f_{bD}$, $\widetilde{f} = f$ and $\widetilde{S} = S$.

# Curvature of surfaces in $\mathbb{R}^n$

Let $D$ be a domain in $\mathbb{R}^2_{(u_1,u_2)}$ and $X = (X_1, \ldots, X_n)\colon D \to \mathbb{R}^n$ be a $\mathscr{C}^2$ embedding. Hence, $S = X(D) \subset \mathbb{R}^n$ is a parametrized surface in $\mathbb{R}^n$.

Every smooth embedded curve in $S$ is of the form

$$\lambda(t) = X(u_1(t), u_2(t)) \in S$$

where $t \mapsto (u_1(t), u_2(t))$ is a smooth embedded curve in $D$.

Let $s = s(t)$ denote the arc length on $\lambda$. The number

$$\kappa(\mathbf{T}, \mathbf{N}) := \frac{d^2\lambda}{ds^2} \cdot \mathbf{N} = \sum_{i,j=1}^{2} \left( X_{u_i u_j} \cdot \mathbf{N} \right) \frac{du_i}{ds} \frac{du_j}{ds}$$

is the **normal curvature** of $S$ at $p = \lambda(t_0) \in S$ in the tangent direction $\mathbf{T} = \lambda'(s_0) \in T_pS$ with respect to the normal vector $\mathbf{N} \in N_pS$.

# Curvature in terms of fundamental forms

In terms of $t$-derivatives we get

$$\kappa(\mathbf{T},\mathbf{N}) = \frac{\sum_{i,j=1}^{2}\left(X_{u_i u_j}\cdot\mathbf{N}\right)\dot{u}_i\dot{u}_j}{\sum_{i,j=1}^{2} g_{i,j}\dot{u}_i\dot{u}_j} = \frac{\text{second fundamental form}}{\text{first fundamental form}}.$$

Fix a normal vector $\mathbf{N}\in N_pS$ and vary the unit tangent vector $\mathbf{T}\in T_pS$. The **principal curvatures** of $S$ at $p$ in direction $\mathbf{N}$ are the numbers

$$\kappa_1(\mathbf{N}) = \max_{\mathbf{T}}\kappa(\mathbf{T},\mathbf{N}), \qquad \kappa_2(\mathbf{N}) = \min_{\mathbf{T}}\kappa(\mathbf{T},\mathbf{N}).$$

Their average

$$\mathrm{H}(\mathbf{N}) = \frac{\kappa_1(\mathbf{N})+\kappa_2(\mathbf{N})}{2}\in\mathbb{R}$$

is the **mean curvature** of $S$ at $p$ in the normal direction $\mathbf{N}\in N_pS$.

# The mean curvature vector

Let $G = (g_{i,j})$ and $b(\mathbf{N}) = (b_{i,j}(\mathbf{N}))$ be the matrix of the 1st and the 2nd fundamental form. The extremal values of $\kappa(\mathbf{T}, \mathbf{N})$ are roots of

$$\det\big(b(\mathbf{N}) - \mu G\big) = 0$$

$$(\det G)\mu^2 - \big(g_{2,2}b_{1,1}(\mathbf{N}) + g_{1,1}b_{2,2}(\mathbf{N}) - 2g_{1,2}b_{1,2}(\mathbf{N})\big)\mu + \det b(\mathbf{N}) = 0.$$

Note that $b_{i,j}(\mathbf{N}) = X_{u_i u_j} \cdot \mathbf{N}$. The Vieta formula gives

$$\mathrm{H}(\mathbf{N}) = \frac{g_{2,2}X_{u_1u_1} + g_{1,1}X_{u_2u_2} - 2g_{1,2}X_{u_1u_2}}{2\det G} \cdot \mathbf{N}.$$

There is a unique normal vector $\mathbf{H} \in N_pS$ such that

$$\mathrm{H}(\mathbf{N}) = \mathbf{H} \cdot \mathbf{N} \quad \text{for all } \mathbf{N} \in N_pS.$$

This $\mathbf{H}$ is the **mean curvature vector** of the surface $S$ at $p$.

# The mean curvature in isothermal coordinates

The formulas simplify drastically in **isothermal coordinates**:

$$(g_{i,j}) = \mu I, \quad \mu = ||X_{u_1 u_1}||^2 = ||X_{u_2 u_2}||^2.$$

$$\mathrm{H}(\mathbf{N}) = \frac{(X_{u_1 u_1} + X_{u_2 u_2})}{2\mu} \cdot \mathbf{N} = \frac{\triangle X}{2\mu} \cdot \mathbf{N}.$$

### Lemma

*Assume that $D$ is a domain in $\mathbb{R}^2_{(u_1,u_2)}$ and $X\colon D \to \mathbb{R}^n$ is a* conformal immersion *of class $\mathscr{C}^2$ (i.e., $u = (u_1, u_2)$ are isothermal for $X$). Then the Laplacian $\triangle X = X_{u_1 u_1} + X_{u_2 u_2}$ is orthogonal to $S = X(D)$ and satisfies*

$$\triangle X = 2\mu \mathbf{H}$$

*where $\mathbf{H}$ is the mean curvature vector and*

$$\mu = ||X_{u_1 u_1}||^2 = ||X_{u_2 u_2}||^2.$$

# Proof of the lemma

It suffices to show that the vector

$$\triangle X(u)$$

is orthogonal to the surface $S$ at the point $X(u)$ for every $u \in D$. If this holds, it follows from the preceding formula that the normal vector

$$\frac{\triangle X(u)}{2\mu} \in N_{X(u)}S$$

fits the definition of the mean curvature vector **H**, so it equals **H**.

Conformality of the immersion $X$ can be written as follows:

$$X_{u_1} \cdot X_{u_1} = X_{u_2} \cdot X_{u_2}, \quad X_{u_1} \cdot X_{u_2} = 0.$$

Differentiating the first identity on $u_1$ and the second one on $u_2$ yields

$$X_{u_1 u_1} \cdot X_{u_1} = X_{u_1 u_2} \cdot X_{u_2} = -X_{u_2 u_2} \cdot X_{u_1},$$

whence $\triangle X \cdot X_{u_1} = 0$. Similarly we get $\triangle X \cdot X_{u_2} = 0$ by differentiating the first identity on $u_2$ and the second one on $u_1$.

# Lagrange's formula for the variation of area

The area of an immersed surface $X\colon D \to \mathbb{R}^n$ with the 1st fundamental form $G = (g_{i,j})$ equals

$$\mathscr{A}(X) = \int_D \sqrt{\det G} \cdot du_1 du_2.$$

Let $\mathbf{N}\colon D \to \mathbb{R}^n$ be a *normal vector field* along $X$ which vanishes in $bD$. Consider the 1-parameter family of maps $X^t\colon D \to \mathbb{R}^n$:

$$X^t(u) = X(u) + t\,\mathbf{N}(u), \quad u \in D,\ t \in \mathbb{R}.$$

A calculation gives the formula for the first variation of area:

$$\delta\mathscr{A}(X)\mathbf{N} = \left.\frac{d}{dt}\right|_{t=0} \mathscr{A}(X^t) = -2\int_D \mathbf{H}\cdot\mathbf{N}\,\sqrt{\det G} \cdot du_1 du_2.$$

It follows that $\qquad \delta\mathscr{A}(X) = 0 \Longleftrightarrow \mathbf{H} = 0.$

# Conformal minimal surfaces are harmonic

In view of the formula $\triangle X = 2\mu\mathbf{H}$ which holds for a conformal immersion $X$ we get the following corollary.

## Corollary

*The following are equivalent for a smooth conformal immersion $X\colon D \to \mathbb{R}^n$ from a domain $D \subset \mathbb{R}^2$:*

- *$X$ is minimal (a stationary point of the area functional)*
- *$X$ has vanishing mean curvature vector:* $\mathbf{H} = 0$
- *$X$ is harmonic:* $\triangle X = 0$

# Minimal surfaces versus area minimizing surfaces

We wish to emphasize the difference between

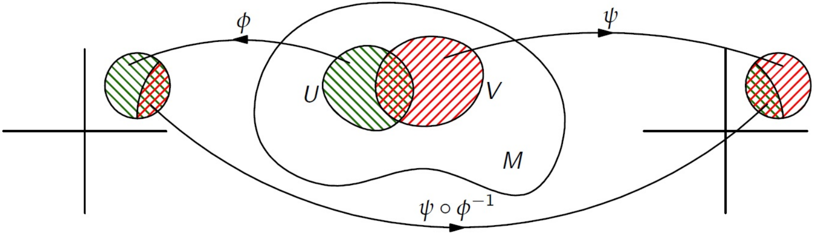
- **minimal surfaces:** these are stationary (critical) points of the area functional, and are only locally area minimizing; and
- **area-minimizing surfaces:** these are surfaces which globally minimize the area among all nearby surfaces with the same boundary.
- Minimal surfaces which are **graphs** are in fact (globally) area minimizing.

# Riemann surfaces

## Definition

A **Riemann surface** $\mathbf{M}$ is a complex manifold of complex dimension one.

So, a Riemann surface $\mathbf{M} = (M, \mathscr{A})$ is nothing but a smooth (real) surface $M$ together with an **atlas** $\mathscr{A}$ such that the change of charts are **biholomorphic functions**.




**Holomorphic functions** $\mathbf{M} \to \mathbb{C}$ and **harmonic functions** $\mathbf{M} \to \mathbb{R}$ on a Riemann surface $\mathbf{M}$ are well defined.

# Riemann surfaces

## Definition

A **compact bordered Riemann surface** $\overline{\mathbf{M}}$ is a compact Riemann surface with nonempty boundary $b\mathbf{M} \neq \emptyset$ consisting of finitely many pairwise disjoint smooth Jordan curves.
The interior $\mathbf{M} = \overline{\mathbf{M}} \setminus b\mathbf{M}$ is said a **bordered Riemann surface**.

Open Riemann surfaces $\mathbf{M}$ (noncompact and with $b\mathbf{M} = \emptyset$) are classified:

- **Hyperbolic**: there are non-constant negative subharmonic functions $\mathbf{M} \to \mathbb{R}$; equivalently, the Brownian motion on $\mathbf{M}$ is transient.

  The unit complex disc $\mathbb{D} \subset \mathbb{C}$ is hyperbolic. Every bordered Riemann surface is hyperbolic.

- **Parabolic**: every negative subharmonic function $\mathbf{M} \to \mathbb{R}$, is constant; equivalently, the Brownian motion on $\mathbf{M}$ is recurrent.

  The complex plane $\mathbb{C}$ is parabolic. Every compact Riemann surface with finitely many points removed is parabolic.

# Riemann surfaces

A Riemann surface $\mathbf{M} = (M, \mathscr{T})$ is the same thing as a smooth (real) oriented surface $M$ together with a **conformal structure** $\mathscr{T}$; i.e., a **conformal class of Riemannian metrics** on $M$. (Two metrics $g_1$, $g_2$ on $M$ are conformal if $g_1 = \lambda g_2$ for some smooth positive function $\lambda$ on $M$.)

**Angles** on a Riemann surface $\mathbf{M}$ are well defined.

## Definition

An **immersion** $\mathbf{X} \colon \mathbf{M} = (M, \mathscr{T}) \to \mathbb{R}^n$ $(n \geq 3)$ is said to be **conformal** if it preserves angles; equivalently, if the Riemannian metric $\mathbf{X}^* ds^2$ induced on $M$ by the Euclidean metric $ds^2$ in $\mathbb{R}^n$ via $\mathbf{X}$ lies in $\mathscr{T}$.

# Conformal minimal surfaces are harmonic

Let $(M, g)$ be a Riemannian surface, let $X : (M, g) \to \mathbb{R}^n$ $(n \geq 3)$ be an **isometric immersion**, and let $H$ be the mean curvature vector of $X$. Then

$$\triangle X = 2H.$$

## Corollary

*The following are equivalent for a conformal immersion* $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n$ $(n \geq 3)$*:*

- $\mathbf{X}$ *is* **minimal** *(i.e., the* **mean curvature vector** *is identically zero).*
- $\mathbf{X}$ *is a* **stationary point of the area functional***.*
- $\mathbf{X}$ *is a* **harmonic map** *(i.e.,* $\triangle \mathbf{X} = 0$ *on* $\mathbf{M}$*).*

In the sequel we shall always assume that $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n$ is conformal and hence

$$\mathbf{X} \text{ is minimal} \iff \delta\mathscr{A}(\mathbf{X}) \iff \triangle \mathbf{X} = 0.$$

# Weierstrass Representation (Osserman 1960s)

Let $\mathbf{M}$ be an open Rieman surface.
Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n) \colon \mathbf{M} \to \mathbb{R}^n$ be a **conformal minimal immersion**.
Let $z$ be a conformal parameter on $\mathbf{M}$ and denote by

$$\phi_j = \partial_z \mathbf{X}_j,\ j = 1, \ldots, n \quad \text{(holomorphic 1-forms in } \mathbf{M}\text{)}.$$

- $\mathbf{X}^* ds^2 = \sum_{j=1}^{n} |\phi_j|^2 \neq 0$ (conformal metric).
- $\Phi = (\phi_1, \ldots, \phi_n)$ satisfies $\sum_{j=1}^{n} \phi_j^2 = 0$.
- Hence $f = \dfrac{\Phi}{\vartheta} \colon \mathbf{M} \to \mathbb{C}^n$ assumes values in the **Null Quadric**:

$$\mathfrak{A}^* = \{\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n \colon \zeta_1^2 + \cdots + \zeta_n^2 = 0\} \setminus \{0\} \subset \mathbb{C}^n,$$

  (here $\vartheta$ is any holomorphic 1-form on $\mathbf{M}$ with no zeros).
- $\Phi$ has no real periods: $\Re \int_\gamma \Phi = 0$ for any closed curve $\gamma \subset \mathbf{M}$.
- $\mathbf{X}(p) = \mathbf{X}(p_0) + \Re \int_{p_0}^{p} \Phi = \mathbf{X}(p_0) + \Re \int_{p_0}^{p} f\vartheta$. **Conversely! (Period Problem)**
- $\Phi \equiv$ **Weierstrass Representation** of $\mathbf{X}$.

# Examples

- The plane: $\mathbf{M} = \mathbb{C}$, $\Phi = (1, -\imath, 0)dz$.
- The catenoid: $\mathbf{M} = \mathbb{C} \setminus \{0\}$, $\Phi = \left(\frac{1}{2}\big(\frac{1}{z^2} - 1\big)\,,\ \frac{\imath}{2}\big(\frac{1}{z^2} + 1\big)\,,\ \frac{1}{z}\right) dz$.
- The helicoid: $\mathbf{M} = \mathbb{C}$, $\Phi = \left(\frac{\imath}{2}\big(e^{-z} - e^{z}\big)\,,\ -\frac{1}{2}\big(e^{-z} + e^{z}\big)\,,\ \imath\right) dz$.

# The Null Quadric is Oka

Let $\vartheta$ be a holomorphic 1-form on $\mathbf{M}$ with no zeros. There is a bijective correspondence (up to translations)

$$\{\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n \text{ CMI}\} \longleftrightarrow \{f\colon \mathbf{M} \to \mathfrak{A}^* \text{ holomorphic, } f\vartheta \text{ no real periods}\}$$

$$\mathbf{X}(p) = \mathbf{X}(p_0) + \Re \int_{p_0}^{p} f\vartheta, \quad f = \frac{\Phi}{\vartheta}.$$

We construct CMIs $\mathbf{M} \to \mathbb{R}^n$ by finding holomorphic maps $f\colon \mathbf{M} \to \mathfrak{A}^* \subset \mathbb{C}^n$ for which $f\vartheta$ has **no real periods**:

$$\Re \int_\Gamma f\vartheta \quad \text{for every closed curve } \Gamma \subset \mathbf{M}.$$

The Null Quadric

$$\mathfrak{A}^* = \{\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n \colon \zeta_1^2 + \cdots + \zeta_n^2 = 0\} \setminus \{0\} \quad (n \geq 3)$$

controlling minimal surfaces in $\mathbb{R}^n$ is an **Oka Manifold**.

# Weierstrass Representation in $\mathbb{R}^3$: the Gauss map

Let $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \colon \mathbf{M} \to \mathbb{R}^3$ be a conformal minimal immersion with Weierstrass representation $\Phi = f\vartheta$, where $f = (\phi_1, \phi_2, \phi_3) \colon \mathbf{M} \to \mathfrak{A}^* \subset \mathbb{C}^3$. Then either $\phi_1 \equiv \imath\phi_2$ and $\mathbf{X}$ is flat, or

$$f = \left( \frac{1}{2}\left(\frac{1}{g} - g\right) \, , \, \frac{\imath}{2}\left(\frac{1}{g} + g\right) \, , \, 1 \right) \phi_3$$

where

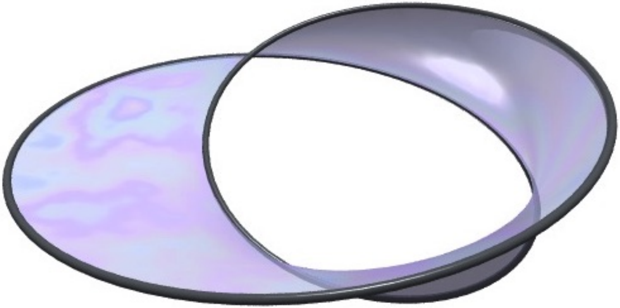
$$g = \frac{\phi_3}{\phi_1 - \imath\phi_2}$$

is the stereographic projection of the Gauss map $\mathbf{M} \to \mathbb{S}^2$ of $\mathbf{X}$.

- $g \colon \mathbf{M} \to \mathbb{C}$ is a meromorphic function, called the **complex Gauss map** of $\mathbf{X}$.
- $\mathbf{X}^* ds^2 = \sum_{j=1}^{n} |\phi_j|^2 = \frac{1}{4}\left(|g| + \frac{1}{|g|}\right)^2 |\phi_3|^2 |\vartheta|^2$. Hence, the zeros and poles of $g$ are zeros of $\phi_3$ with the same order.
- $(g, \phi_3\theta) \equiv$ **Weierstrass Representation** of $\mathbf{X}$. Conversely! (Period Problem)

# Nonorientable minimal surfaces

**Nonorientable** surfaces present themselves quite naturally in the very origin of Minimal Surface Theory.



# Nonorientable minimal surfaces

Assume that $\mathbf{M}$ carries an antiholomorphic involution $\mathfrak{I}\colon \mathbf{M} \to \mathbf{M}$ without fixed points. If $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n$ is a conformal minimal immersion such that

$$\mathbf{X} \circ \mathfrak{I} = \mathbf{X}, \tag{3}$$

then $\mathbf{X}(\mathbf{M}) \subset \mathbb{R}^n$ is a **nonorientable** minimal surface.

If $\Phi = f\vartheta$ are the Weierstrass data of $\mathbf{X}$ and $\mathfrak{I}^*\vartheta = \overline{\vartheta}$, then (3) is equivalent to

$$f \circ \mathfrak{I} = \overline{f}.$$

If $n = 3$, $f = (\phi_1, \phi_2, \phi_3)$, and $g = \dfrac{\phi_3}{\phi_1 - \imath\phi_2}$, then (3) is equivalent to

$$\phi_3 \circ \mathfrak{I} = \overline{\phi}_3, \quad g \circ \mathfrak{I} = -\frac{1}{\overline{g}}.$$

# Holomorphic null curves in $\mathbb{C}^n$

## Definition

Let $\mathbf{M}$ be a Riemann surface. A holomorphic immersion

$$\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n) \colon \mathbf{M} \to \mathbb{C}^n$$

is a **null curve** if the derivative $\mathbf{F}' = (\mathbf{F}_1', \mathbf{F}_2', \ldots, \mathbf{F}_n')$ with respect to any local holomorphic coordinate $\zeta = x + \imath y$ on $\mathbf{M}$ satisfies

$$(\mathbf{F}_1')^2 + (\mathbf{F}_2')^2 + \ldots + (\mathbf{F}_n')^2 = 0,$$

The nullity condition is equivalent to

$$\mathbf{F}'(\zeta) \in \mathfrak{A}^* = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^{n} z_j^2 = 0\},$$

where $\mathfrak{A}^*$ is the **null quadric** in $\mathbb{C}^n$.

# Holomorphic null curves in $\mathbb{C}^n$

Let $\vartheta$ be a holomorphic 1-form on $\mathbf{M}$ with no zeros. There is a bijective correspondence (up to translations)

$$\big\{\mathbf{F}\colon \mathbf{M} \to \mathbb{C}^n \text{ null curves}\big\} \longleftrightarrow \big\{f\colon \mathbf{M} \to \mathfrak{A}^* \text{ holomorphic, } f\vartheta \text{ exact}\big\}$$

$$\mathbf{F}(p) = \mathbf{F}(p_0) + \int_{p_0}^{p} f\vartheta, \quad f = \frac{\Phi}{\vartheta}.$$

We construct null curves $\mathbf{M} \to \mathbb{C}^n$ by finding holomorphic maps $f\colon \mathbf{M} \to \mathfrak{A}^* \subset \mathbb{C}^n$ for which $f\vartheta$ is an **exact 1-form**:

$$\int_\Gamma f\vartheta \quad \text{for every closed curve } \Gamma \subset \mathbf{M}.$$

# Connection between null curves and minimal surfaces

If $\mathbf{F} = \mathbf{X} + \imath\mathbf{Y} \colon \mathbf{M} \to \mathbb{C}^n$ is a holomorphic null curve, then

$$\mathbf{X} = \Re\mathbf{F} \colon \mathbf{M} \to \mathbb{R}^n, \quad \mathbf{Y} = \Im\mathbf{F} \colon \mathbf{M} \to \mathbb{R}^n$$

are conformal harmonic (hence minimal) immersions into $\mathbb{R}^n$.

Conversely, a conformal minimal immersion $\mathbf{X} \colon \mathbb{D} \to \mathbb{R}^n$ of the disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ is the real part of a holomorphic null curve $\mathbf{F} \colon \mathbb{D} \to \mathbb{C}^n$. (This fails on multiply connected Riemann surfaces.)

If $\mathbf{F} = \mathbf{X} + \imath\mathbf{Y} \colon \mathbf{M} \to \mathbb{C}^n$ is a null curve then

$$\mathbf{F}^* ds^2_{\mathbb{C}^n} = 2\mathbf{X}^* ds^2_{\mathbb{R}^n} = 2\mathbf{Y}^* ds^2_{\mathbb{R}^n}.$$

# The calculation

Let $\mathbf{F} = \mathbf{X} + \imath \mathbf{Y} = (\mathbf{F}^1, \ldots, \mathbf{F}^n) \colon \mathbf{M} \to \mathbb{C}^n$ be a holomorphic null curve and $\zeta = x + \imath y$ a local holomorphic coordinate on $\mathbf{M}$. Then

$$\begin{aligned} 0 &= \sum_{j=1}^n (\mathbf{F}^j_\zeta)^2 = \sum_{j=1}^n (\mathbf{F}^j_x)^2 = \sum_{j=1}^n \left(\mathbf{X}^j_x + \imath \mathbf{Y}^j_x\right)^2 \\ &= \sum_{j=1}^n \left((\mathbf{X}^j_x)^2 - (\mathbf{Y}^j_x)^2\right) + 2\imath \sum_{j=1}^n \mathbf{X}^j_x \mathbf{Y}^j_x. \end{aligned}$$

Since $\mathbf{Y}_x = -\mathbf{X}_y$ by the Cauchy-Riemann equations,

$$0 = |\mathbf{X}_x|^2 - |\mathbf{X}_y|^2 - 2\imath\, \mathbf{X}_x \cdot \mathbf{X}_y \iff |\mathbf{X}_x| = |\mathbf{X}_y|,\ \mathbf{X}_x \cdot \mathbf{X}_y = 0.$$

It follows that $\mathbf{X}$ is conformal harmonic and

$$\mathbf{F}^* ds^2_{\mathbb{C}^n} = |\mathbf{F}_x|^2 (dx^2 + dy^2) = 2|\mathbf{X}_x|^2 (dx^2 + dy^2) = 2\mathbf{X}^* ds^2_{\mathbb{R}^n} = 2\mathbf{Y}^* ds^2_{\mathbb{R}^n}.$$

# Example: catenoid and helicoid

The **catenoid** and the **helicoid** are **conjugate minimal surfaces**; i.e., the real and the imaginary part of the same null curve

$$\mathbf{F}(\zeta) = (\cos\zeta, \sin\zeta, -\imath\zeta), \qquad \zeta = x + \imath y \in \mathbb{C}.$$

Consider the family of conformal minimal immersions ($t \in \mathbb{R}$):

$$\begin{aligned} \mathbf{X}_t(\zeta) &= \Re\left(e^{\imath t}\mathbf{F}(\zeta)\right) \\ &= \cos t \begin{pmatrix} \cos x \cdot \cosh y \\ \sin x \cdot \cosh y \\ y \end{pmatrix} + \sin t \begin{pmatrix} \sin x \cdot \sinh y \\ -\cos x \cdot \sinh y \\ x \end{pmatrix} \end{aligned}$$

At $t = 0$ we have a catenoid and at $t = \pm\pi/2$ a helicoid.

Two minimal surfaces are said to be **associated** to each other if they are obtained as projection of the same null curve.

# The flux of a conformal minimal immersion

The **flux map** of a conformal minimal immersion $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n$ is the group homomorphism

$$\mathrm{Flux}_{\mathbf{X}}\colon H_1(\mathbf{M};\mathbb{Z}) \to \mathbb{R}^n$$

given on any closed curve $C$ in $\mathbf{M}$ by

$$\mathrm{Flux}_{\mathbf{X}}(C) = \int_C \Im\Phi = \int_C d^c\mathbf{X}$$

where $\Phi = \partial_z \mathbf{X}$ are the Weierstrass data of $\mathbf{X}$ and

$$d^c\mathbf{X} = \imath(\overline{\partial}\mathbf{X} - \partial\mathbf{X})$$

is **the conjugate differential**. (Note that $d^c\mathbf{X}$ is closed precisely when $\mathbf{X}$ is harmonic: $dd^c\mathbf{X} = 0$.)

A conformal minimal immersion $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n$ is the real part of a holomorphic null curve $\mathbf{M} \to \mathbb{C}^n$ if and only if $\mathrm{Flux}_{\mathbf{X}} = 0$.

# Global Theory

Let $\mathbf{X} : \mathbf{M} \to \mathbb{R}^n$ $(n \geq 3)$ be a conformal minimal immersion.

## Definition

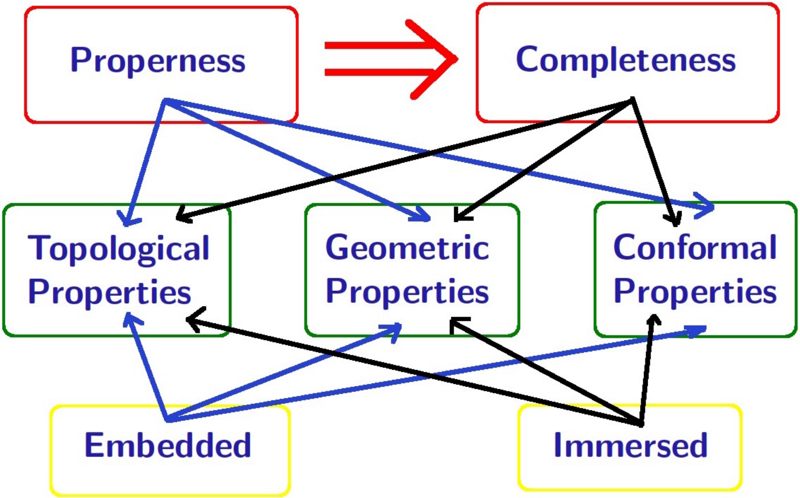
$\mathbf{X}$ (or $\mathbf{X}(\mathbf{M})$) is said to be **complete** if the induced metric $\mathbf{X}^* ds^2 = \sum_{j=1}^{n} |\phi_j|^2$ is a complete Riemannian metric on $\mathbf{M}$; equivalently if the Euclidean length of $\mathbf{X}(\gamma) \subset \mathbb{R}^n$ is infinite for any divergent curve $\gamma \subset \mathbf{M}$.

## Definition

$\mathbf{X}$ (or $\mathbf{X}(\mathbf{M})$) is said to be **proper** if $\mathbf{X}(\gamma) \subset \mathbb{R}^n$ diverges for any divergent curve $\gamma \subset \mathbf{M}$.

Properness trivially implies completeness.

# Global Theory



# Finite Total Curvature

$\mathbf{X} : \mathbf{M} \to \mathbb{R}^3$ is said to be of **finite total curvature** if

$$\int_{\mathbf{M}} |K| = -\int_{\mathbf{M}} K < +\infty,$$

where $K \leq 0$ denotes the Gauss curvature function of $\mathbf{X}$.

1963 **Osserman** A conformal complete minimal immersion $\mathbf{X} \colon \mathbf{M} \to \mathbb{R}^3$ is of FTC if and only if

- $\mathbf{M} \cong \overline{\mathbf{M}} \setminus E$, where $\overline{\mathbf{M}}$ is a compact Riemann surface and $E \subset \overline{\mathbf{M}}$ is finite; that is, of **finite topology** and **parabolic conformal type**.
- The Weierstrass data $\phi_j = \partial_z \mathbf{X}_j$ (and so $g$) extend meromorphically to $\overline{\mathbf{M}}$.

The wealth of examples, discovered since about 1980, rely on this theorem.
Very well studied theory: General existence results, classification theorems, asymptotic behavior, behavior of the Gauss map,...

1983 **Jorge-Meeks** Every complete minimal surface in $\mathbb{R}^3$ of FTC is **proper** in $\mathbb{R}^3$.

# Completeness, Properness, and Conformal Structure

By the Maximum Principle for harmonic maps, there is no **compact** minimal surface in $\mathbb{R}^3$.
All the complete minimal surfaces in $\mathbb{R}^3$ discovered up to the third quarter of the 19th century are **proper** and (except a few periodic ones) of **parabolic** conformal type.

1963 **Calabi's Conjecture** Every **complete** minimal surface in $\mathbb{R}^3$ is **unbounded**. (Disproved by **Nadirashvili** in **1996**.)

1980s **Sullivan's Conjecture** Every **proper** minimal surface in $\mathbb{R}^3$ of **finite topology** is **parabolic**. (Disproved by **Morales** in **2003**.)

1985 **Schoen-Yau's Conjecture** Every minimal surface in $\mathbb{R}^3$ **properly projecting** into $\mathbb{R}^2$ is **parabolic**. (Disproved by **AA-López** in **2012**.)

# Completeness, Properness, and Conformal Structure

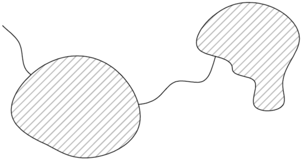
## Question

*How to construct complete/proper minimal surfaces in* $\mathbb{R}^3$ *with* **hyperbolic** *conformal type?*
*Even more, with* **given conformal structure***?*

# Mergelyan Theorem for conformal minimal immersions

## Theorem

*Let* $\mathbf{M}$ *be an open Riemann surface and let* $K \subset \mathbf{M}$ *be a* Runge *compact subset (consisting of finitely many pairwise disjoint, compact, smoothly bounded domains and smooth arcs).*
*Every* **smooth immersion** $K \to \mathbb{R}^n$ $(n \geq 3)$ *being a conformal minimal immersion in* $\overline{\mathring{K}}$ *can be uniformly approximated on* $K$ *by* **conformal minimal immersions** $\mathbf{M} \to \mathbb{R}^n$.



[A. Alarcón, F.J. López: *Minimal surfaces in* $\mathbb{R}^3$ *properly projecting into* $\mathbb{R}^2$. J. Differential Geom. 2012]
[A. Alarcón, F. Forstnerič, F.J. López: *Embedded minimal surfaces in* $\mathbb{R}^n$. Preprint 2014]

# General position of minimal surfaces

Self-intersections of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ are **stable** under deformations.

### Theorem

*Let $n \in \mathbb{N}$, $n \geq 5$, and let $\mathbf{M} = \mathring{\mathbf{M}} \cup b\mathbf{M}$ be a compact bordered Riemann surface.*
*Every conformal minimal immersion $\mathbf{X} \colon \mathbf{M} \to \mathbb{R}^n$ can be uniformly approximated in the $\mathscr{C}^0(\mathbf{M})$-topology by conformal minimal* **embeddings** $\mathbf{Y} \colon \mathbf{M} \to \mathbb{R}^n$.

### Corollary

*Let $n \in \mathbb{N}$, $n \geq 5$, and let $\mathbf{M}$ be an open Riemann surface.*
*Every conformal minimal immersion $\mathbf{X} \colon \mathbf{M} \to \mathbb{R}^n$ can be uniformly approximated in compact subsets of $\mathbf{M}$ by conformal minimal* **embeddings** $\mathbf{Y} \colon \mathbf{M} \to \mathbb{R}^n$.

[A. Alarcón, F. Forstnerič, F.J. López: *Embedded minimal surfaces in $\mathbb{R}^n$.* Preprint 2014]

# Schoen-Yau's conjecture and the embedding theorem

**Theorem**

*Let $n \in \mathbb{N}$, $n \geq 3$, and let $\mathbf{M}$ be an open Riemann surface. There is a conformal minimal immersion $(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n) \colon \mathbf{M} \to \mathbb{R}^n$, embedding if $n \geq 5$, such that $(\mathbf{Y}_1, \mathbf{Y}_2) \colon \mathbf{M} \to \mathbb{R}^2$ is a proper map.*

[A. Alarcón, F.J. López: *Minimal surfaces in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$*. J. Differential Geom. 2012]
[A. Alarcón, F. Forstnerič, F.J. López: *Embedded minimal surfaces in $\mathbb{R}^n$*. Preprint 2014]

# Schoen-Yau's conjecture

## Corollary

*Every open Riemann surface is the underlying conformal structure of a minimal surface in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$.*

1980s **Sullivan's Conjecture** Every **proper** minimal surface in $\mathbb{R}^3$ of **finite topology** is **parabolic**. (Disproved by **Morales** in **2003**.)

1985 **Schoen-Yau's Conjecture** Every minimal surface in $\mathbb{R}^3$ **properly projecting** into $\mathbb{R}^2$ is **parabolic**.

# The embedding theorem

1950s **Bishop, Narasimhan, Remmert** Every open Riemann surface properly embeds in $\mathbb{C}^3$ as a complex curve, hence in $\mathbb{R}^6$ as a conformal minimal surface.

1975 **Greene-Wu** Every open Riemann surface properly embeds in $\mathbb{R}^5$ by harmonic functions.

### Corollary

*Every open Riemann surface is the underlying conformal structure of a properly embedded minimal surface in $\mathbb{R}^5$.*

False in $\mathbb{R}^3$.

Open in $\mathbb{R}^4$. (Examples with arbitrary orientable **topology** are known. Related to the **Bell-Narasimhan** conjecture.)

# Null curves

We may prescribed the flux of the examples in all the previous results.

These results were inspired by similar ones for null holomorphic curves and, more generally, directed holomorphic immersions of open Riemann surfaces into $\mathbb{C}^n$ when the directing variety is Oka were proved in

[A. Alarcón, F. Forstnerič: *Null curves and directed immersions of open Riemann surfaces*. Invent. Math. (2014)]

In the holomorphic case, the general position is embedded in $\mathbb{C}^3$.

# $h$-Runge approximation for conformal minimal immersions

### Theorem

*Let $n \geq 3$. Let* $\mathbf{M}$ *be an open Riemann surface and let $K \subset \mathbf{M}$ be a* Runge *compact subset (consisting of finitely many pairwise disjoint, compact, smoothly bounded domains and smooth arcs). Let* $\mathbf{X}_t \colon K \to \mathbb{R}^n$ $(t \in [0,1])$ *be a* **smooth homotopy** *of smooth immerions such that $\mathbf{X}_t$ is a conformal minimal immersion in $\mathring{K}$ for all $t$ and $\mathbf{X}_0$ is a conformal nonflat minimal immersion on $\mathbf{M}$.*
*Then the homotopy $\mathbf{X}_t$ can be uniformly approximated on $K$ by a smooth homotopy of* **conformal minimal immersions** $\mathbf{Y}_t \colon \mathbf{M} \to \mathbb{R}^n$ *such that*

- $\mathbf{Y}_0 = \mathbf{X}_0$.
- $\mathrm{Flux}_{\mathbf{Y}_t} = \mathrm{Flux}_{\mathbf{X}_t}$ *on $H_1(K, \mathbb{Z})$.*
- $\mathrm{Flux}_{\mathbf{Y}_1}$ *can be prescribed on $H_1(\mathbf{M}, \mathbb{Z}) \setminus H_1(K, \mathbb{Z})$.*

[A. Alarcón, F. Forstnerič: *Every conformal minimal surface in $\mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve.* J. Reine Angew. Math. (Crelle), in press]

# Homotopies

## Theorem

*Let* $\mathbf{M}$ *be an open Riemann surface and* $n \geq 3$*.*

- *For every conformal minimal immersion* $\mathbf{X}_0 \colon \mathbf{M} \to \mathbb{R}^n$ *there exists a smooth homotopy* $\mathbf{X}_t \colon \mathbf{M} \to \mathbb{R}^n$ $(t \in [0,1])$ *of conformal minimal immersions such that* $\mathbf{X}_t$ *is the real part of a null curve* $\mathbf{M} \to \mathbb{C}^n$*.*
- *If* $\mathbf{X}_0$ *is nonflat and* $\mathfrak{p} \colon H_1(\mathbf{M};\mathbb{Z}) \to \mathbb{R}^n$ *is a group homomorphism then there exists a smooth homotopy* $\mathbf{X}_t \colon \mathbf{M} \to \mathbb{R}^n$ $(t \in [0,1])$ *of conformal minimal immersions such that*

$$\mathbf{X}_1 \textit{ is complete and } \mathrm{Flux}_{\mathbf{X}_1} = \mathfrak{p}.$$

- *If* $\mathbf{X}_0$ *is complete then we can choose* $\mathbf{X}_t$ *(as above) to be complete for every* $t \in [0,1]$*.*

[A. Alarcón, F. Forstnerič: *Every conformal minimal surface in* $\mathbb{R}^3$ *is isotopic to the real part of a holomorphic null curve.* J. Reine Angew. Math. (Crelle), in press]

# Riemann-Hilbert method for conformal minimal immersions

### Theorem

*Let* $\mathbf{M}$ *be a compact bordered Riemann surface and let* $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n$ $(n \geq 3)$ *be a conformal minimal immersion (the* **central surface***).*
*Let* $I$ *be a compact subarc of* $b\mathbf{M}$ *which is not a connected component of* $b\mathbf{M}$*. Choose a small annular neighborhood* $A \subset \mathbf{M}$ *of the component* $C$ *of* $b\mathbf{M}$ *containing* $I$ *and a smooth retraction* $\rho\colon A \to C$*.*
*Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *be a couple of unitary orthogonal vectors (the* **direction vectors***), let* $\mu\colon C \to \mathbb{R}_+$ *be a continuous function supported on* $I$ *(the* **size function***), and consider the continuous map*

$$\varkappa\colon b\mathbf{M} \times \overline{\mathbb{D}} \to \mathbb{R}^n$$

$$\varkappa(x,\xi) = \begin{cases} \mathbf{X}(p); & p \in b\mathbf{M} \setminus I \\ \mathbf{X}(p) + \mu(p)\big(\Re\xi\mathbf{u} + \Im\xi\mathbf{v}\big), & p \in I. \end{cases}$$

# Riemann-Hilbert method for conformal minimal immersions

### Theorem (Continued)

*Then for any number $\epsilon > 0$ there exist an arbitrarily small open neighborhood $\Omega$ of $I$ in $A$ and a conformal minimal immersion $\mathbf{Y}\colon \mathbf{M} \to \mathbb{R}^n$ satisfying the following properties:*

- $\mathbf{Y}$ *is $\epsilon$-close to $\mathbf{X}$ in the $\mathcal{C}^1$ topology on $\mathbf{M} \setminus \Omega$.*
- $\mathrm{dist}(\mathbf{Y}(p), \varkappa(p, b\mathbb{D})) < \epsilon$ *for all $p \in b\mathbf{M}$.*
- $\mathrm{dist}(\mathbf{Y}(p), \varkappa(\rho(p), \overline{\mathbb{D}})) < \epsilon$ *for all $p \in \Omega$.*

[A. Alarcón, F Forstnerič: *The Calabi-Yau problem, null curves, and Bryant surfaces.* Math. Ann., in press]
[A. Alarcón, B. Drinovec, F. Forstnerič, F.J. López: *Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves.* Proc. London Math. Soc., in press]
[A. Alarcón, B. Drinovec, F. Forstnerič, F.J. López: *Minimal surfaces in minimally convex domains.* In preparation]

We do not change the conformal structure on $\mathbf{M}$.

# Calabi-Yau Problem

1963 **Calabi's Conjecture** Every **complete** minimal surface in $\mathbb{R}^3$ is **unbounded**. (Disproved by **Nadirashvili** in **1996**.)

2000 **Yau** What is the geometry of complete bounded minimal surfaces?

2006 **Martín-Morales** There are complete proper minimal surfaces in any convex domain of $\mathbb{R}^3$.

**What about domains other than convex ones?**

2012 **Ferrer-Martín-Meeks** There are complete bounded minimal surfaces in $\mathbb{R}^3$ with arbitrary topology.

**What about the conformal structure?**

# Calabi-Yau Problem

**Theorem**

*Let $n \in \mathbb{N}$, $n \geq 3$, and let $\mathcal{D}$ be a convex domain in $\mathbb{R}^n$. Let* $\mathbf{M}$ *be a bordered Riemann surface and let* $\mathbf{X}\colon \overline{\mathbf{M}} \to \mathbb{R}^n$ *be a conformal minimal immersion with* $\mathbf{X}(\overline{\mathbf{M}}) \subset \mathcal{D}$.
$\mathbf{X}$ *may be approximated uniformly in compacta in* $\mathbf{M}$ *by complete proper conformal minimal immersions (embeddings if $n \geq 5$)* $\mathbf{M} \to \mathcal{D}$.

**Theorem**

*Every compact bordered Riemann surface* $\mathbf{M}$ *is the underlying conformal structure of complete minimal surface in $\mathbb{R}^3$ bounded by Jordan curves.*

[A. Alarcón, F Forstnerič: *The Calabi-Yau problem, null curves, and Bryant surfaces.* Math. Ann., in press]
[A. Alarcón, B. Drinovec, F. Forstnerič, F.J. López: *Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves.* Preprint 2015]

# Calabi-Yau Problem

**Theorem**

*Let $\mathcal{D}$ be a mean-convex domain in $\mathbb{R}^3$.*
*Every bordered Riemann surface* **M** *carries a complete proper conformal minimal immersion* $\mathbf{M} \to \mathcal{D}$.

[A. Alarcón, B. Drinovec, F. Forstnerič, F.J. López: *Minimal surfaces in minimally convex domains.* In preparation]

These results contribute to the **conformal** and **asymptotic** Calabi-Yau problems.

# Minimal Surfaces and Complex Analysis

## Lecture 1: Survey

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