# CR Geometry, Mappings into Spheres, and Sums-Of-Squares
# Lecture IV-V

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October 1, 2015

# Outline - Lectures IV-V


1. Levi Nondegenerate Hypersurfaces in $\mathbb{C}^{n+1}$
2. Chern-Moser Normal Form
3. E. Cartan's Approach to CR Geometry
4. Pseudohermitian Geometry
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# Levi nondegenerate hypersurfaces in $\mathbb{C}^{n+1}$.

Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface, and $p \in M$.

**Definitions.**

- $M$ is Levi nondegenerate at $p$ if the Levi form

$$\mathcal{L}_p^\theta \colon T_p^{1,0}M \times T_p^{1,0}M \to \mathbb{C}$$

at $p$ is nondegenerate for some (and hence all) contact forms $\theta$.

- $M$ is strictly pseudoconvex at $p$ if $\mathcal{L}_p^\theta$ is (positive) definite.

Fix $p \in M$. Choose local coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ such that

$$p = (0,0), \quad T_0^{0,1}M = \{w = 0\}, \quad T_0 M = \{\operatorname{Im} w = 0\}.$$

Express $M$ in graph form:

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w), \quad \phi(0) = 0,\ d\phi(0) = 0;\ \phi \in C^\kappa.$$

# The Levi form.

A computation (see Lecture I) shows that the Levi form $\mathcal{L}_0^\theta$, with $\theta = i\partial\rho|_M$, is represented by

$$\mathcal{L}_0(a, \bar{a}) = \sum_{j,k=1}^{n} \frac{\partial\phi}{\partial z_j \bar{z}_k}(0) a_j \bar{a}_k, \quad a \in T_0^{1,0} M \cong \mathbb{C}^n.$$

**Assume:** $M$ is Levi nondegenerate at 0; i.e.,

$$\det(\phi_{z_j \bar{z}_k}(0))_{j,k=1}^n \neq 0.$$

A linear change $(z, w) \mapsto (Az, \pm w)$, $A \in GL(\mathbb{C}^n)$, will make

$$(\phi_{z_j \bar{z}_k}(0))_{j,k=1}^n = I_\ell,$$

where $I_\ell =$ diagonal matrix $D(-1, \ldots, -1, +1, \ldots, +1)$, with $\ell$ "-1" and $n - \ell$ "+1" for some $0 \leq \ell \leq n/2$. $\ell$ is called signature of $M$.

# The quadric $Q^n_\ell$ and weights.

A polynomial change $(z, w) \mapsto (z, w - p(z))$, with $p(z)$ suitable quadratic polynomial, yields

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w) = \langle z, \bar{z} \rangle_\ell + O_{wt}(3), \tag{1}$$

where

$$\langle z, \zeta \rangle_\ell := -\sum_{j=1}^{\ell} z_j \zeta_j + \sum_{j=\ell+1}^{n} z_j \zeta_j$$

and we assign weights $\operatorname{wt} z = 1$, $\operatorname{wt} w = 2$. The quadric $Q^n_\ell$ is the model

$$\operatorname{Im} w = \langle z, \bar{z} \rangle_\ell.$$

# Automorphisms of the model $Q^n_\ell$.

The stability group $\mathrm{Aut}_0(Q^n_\ell)$ consists of:

$$(z,w) \mapsto \left( \frac{\lambda(z-aw)U}{1-2izI_\ell a^* - (r+iaI_\ell a^*)w}, \frac{\sigma\lambda^2 w}{1-2izI_\ell a^* - (r+iaI_\ell a^*)w} \right),$$

where $\lambda > 0$, $a \in \mathbb{C}^n$, $r \in \mathbb{R}$, $\sigma = \pm 1$, and

$$U^* I_\ell U = \sigma I_\ell.$$

## Proposition 1

Any biholomorphism $\Phi(z,w)$, with $\Phi(0)=0$ and preserving the form (1) of $M$, factors uniquely as $\Phi = H \circ \Phi_0$, with $\Phi_0 \in \mathrm{Aut}_0(Q^n_\ell)$ and

$$H(z,w) = (z + f(z,w), w + g(z,w)),$$

where

$$(f(0),\ df(0),\ g(0),\ dg(0),\ g_{z_j z_k}(0),\ \mathrm{Re}\, g_{w^2}(0)) = 0. \tag{2}$$

# Decomposition of power series by type.

Let $F(z,\bar{z},s)$ be a formal power series. $F$ is said to be of type $(k,l)$ if

$$F(rz,t\bar{z},s)=r^kt^lF(z,\bar{z},s),$$

and is then a polynomial in $z$ and $\bar{z}$. Any $F(z,\bar{z},s)$ can be decomposed into type as

$$F(z,\bar{z},s)=\sum_{k,l\geq 0}F_{kl}(z,\bar{z},s),$$

where $F_{kl}(z,\bar{z},s)$ has type $(k,l)$. $F(z,\bar{z},s)$ is Hermitian (real) if

$$F_{lk}(z,\bar{z},s)=\overline{F_{kl}(z,\bar{z},s)}.$$

# The trace operator Tr .

If $F_{kl}(z, \bar{z}, s)$ has type $(k, l)$, then it has "tensor form"

$$F_{kl}(z, \bar{z}, s) = a_{\alpha_1 \dots \alpha_k, \bar{\beta}_1 \dots \bar{\beta}_l}(s) z^{\alpha_1} \dots z^{\alpha_k} \overline{z^{\beta_1}} \dots \overline{z^{\beta_l}},$$

where $z = (z^1, \dots, z^n), \quad \alpha_i, \beta_j = 1, \dots, n.$ We shall write

$$\langle z, \bar{z} \rangle_\ell = h_{\alpha\bar{\beta}} z^\alpha \overline{z^\beta}.$$

The trace of $F_{kl}(z, \bar{z}, s)$ is of type $(k-1, l-1)$, defined by

$$\operatorname{Tr} F_{kl}(z, \bar{z}, s) = b_{\alpha_1 \dots \alpha_{k-1}, \bar{\beta}_1 \dots \bar{\beta}_{l-1}} z^{\alpha_1} \dots z^{\alpha_{k-1}} \overline{z^{\beta_1}} \dots \overline{z^{\beta_{l-1}}},$$

where

$$b_{\alpha_1 \dots \alpha_{k-1}, \bar{\beta}_1 \dots \bar{\beta}_{l-1}} = h^{\gamma\bar{\mu}} a_{\alpha_1 \dots \alpha_{k-1}\gamma, \bar{\beta}_1 \dots \bar{\beta}_{l-1}\bar{\mu}}, \quad h^{\alpha\bar{\mu}} h_{\beta\bar{\mu}} = \delta^\alpha{}_\beta.$$

# Chern-Moser normal form [3].

## Theorem CM-1

Let $M$ be given by (1). Then, there is a unique formal transformation of the form

$$(z, w) \mapsto (z + f(z, w), w + g(z, w)),$$

where $f, g$ satisfy the normalization (2), such that $M$ is given by

$$\operatorname{Im} w = \langle z, \bar{z} \rangle_\ell + N(z, \bar{z}, \operatorname{Re} w), \tag{3}$$

where $N(z, \bar{z}, s)$ is in Chern-Moser normal form:

$$\begin{aligned} N_{kl}(z, \bar{z}, s) &= 0, \quad \min(k, l) \leq 1; \\ \operatorname{Tr} N_{22}(z, \bar{z}, s) = (\operatorname{Tr})^2 N_{32}(z, \bar{z}, s) &= (\operatorname{Tr})^3 N_{33}(z, \bar{z}, s) = 0. \end{aligned} \tag{4}$$

**Remark.** For a given $M$, the space $\mathrm{Aut}_0(Q_\ell^n)$ acts on the space of CM normal forms by Proposition 1.

# Real-analytic hypersurfaces and geometry.

**Theorem CM-2**

If $M$ is $C^\omega$, then the unique transformation to normal form in Theorem CM-1 is convergent, i.e., a biholomorphism.

- The first set of equations in (4) corresponds to transforming a given framed, transverse curve $(\gamma, e_\alpha)\colon (-\epsilon, \epsilon) \to T^{1,0}M$ into

$$(\gamma(t), e_\alpha(t)) = ((0, t), \partial/\partial z^\alpha).$$

- The second set is a system of ODEs (of order 3) for the framed curve. The initial data consist of a direction for $\gamma$ at 0, an orthonormal basis $\{e_\alpha\}$ at 0 for $T_0^{1,0}M$, and a real parameter fixing the parameterization; these initial conditions are parametrized by $\mathrm{Aut}_0(Q^n_\ell)$.
- The curves $\gamma$ that yield solutions to this system of ODEs are called chains. These are important geometric objects associated with $M$.

# The CR curvature $S = (S_{\alpha\bar{\beta}\nu\bar{\mu}})$.

The Levi form provides a first, very rough classification of Levi nondegenerate hypersurfaces $M \subset \mathbb{C}^{n+1}$ via the signature $\ell$. The next interesting invariant is the CR curvature, defined as follows:

**Definition.** If $M$ is given at $p \in M$ in normal form (3) and (4), then the CR curvature of $M$ at $p$ is $S_{\alpha\bar{\beta}\nu\bar{\mu}}$, where $N_{22}(z,\bar{z},0)$ is given in tensor form:

$$N_{22}(z,\bar{z},0) = S_{\alpha\bar{\beta}\nu\bar{\mu}} z^{\alpha} z^{\nu} \overline{z^{\beta} z^{\mu}}. \tag{5}$$

**Remarks.** Recall that $\operatorname{Tr} N_{22} = 0 \implies S_{\alpha\bar{\beta}\nu}{}^{\nu} := h^{\nu\bar{\mu}} S_{\alpha\bar{\beta}\nu\bar{\mu}} = 0$. For $n = 1$ (i.e., in $\mathbb{C}^2$), this means $S_{\alpha\bar{\beta}\nu\bar{\mu}} = 0$, so CR curvature is only interesting when $n \geq 2$. In $\mathbb{C}^2$, the interesting invariant is E. Cartan's "6th order tensor".

- For $n \geq 2$, $M$ is locally "spherical" (equivalent to quadric) $\iff$ $S_{\alpha\bar{\beta}\nu\bar{\mu}} \equiv 0$.

E. Cartan's approach

# CR coframes on a CR manifold (hypersurface type).

Let $M$ be a $2n+1$-dimensional CR manifold;

- CR bundle $T^{0,1}M$, CR-dim $M = n$.

In an open subset $U \subset M$:

- Fix a contact form $\theta$ on $M$; $\iff$ $\theta$ is real and

$$\theta^{\perp} = T^{1,0}M \oplus T^{0,1}M.$$

- Add linearly independent 1-forms $\theta^1, \ldots, \theta^n$ such that

$$(\theta, \theta^1, \ldots \theta^n)^{\perp} = T^{0,1}M.$$

- Set $\theta^{\bar\alpha} = \overline{\theta^\alpha}$; Convention: $\alpha, \beta, \ldots = 1, \ldots, n$.
- $(\theta, \theta^\alpha, \theta^{\bar\beta})$ is coframe for $M$ in $U$; $(\theta, \theta^\alpha)$ is called a CR coframe.

# Change of coframe and CTCM coframes.

Any other CR coframe $(\tilde{\theta}, \tilde{\theta}^\alpha)$ in $U \subset M$ must be of the form

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\theta}^\alpha \end{pmatrix} = \begin{pmatrix} u & 0 \\ u^\alpha & {u_\beta}^\alpha \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\beta \end{pmatrix}.$$

For a choice of CR coframe $(\theta, \theta^\alpha)$,

$$d\theta = i h_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}} + \theta \wedge \phi_0, \tag{6}$$

where $h_{\alpha\bar{\beta}}$ is the Levi form $\mathcal{L}^\theta(L_\alpha, L_\beta)$ and $\phi_0$ a real 1-form, determined only up to $\phi_0 \mapsto \phi_0 + v\theta$.

**Definition.** A choice of $(\theta, \theta^\alpha, \theta^{\bar{\beta}}, \phi_0)$ (as above) is called a CTCM coframe.

CTCM = Cartan-Tanaka-Chern-Moser, [1, 2, 4, 3].

# First prolongation; the bundle of contact forms $E \to M$.

Let $E \to M$ be the $\mathbb{R}_+$ bundle of contact forms such that the Levi form $h_{\alpha\bar{\beta}}$ has $\ell \leq n/2$ negative eigenvalues. For a fixed such $\theta$ and $x \in M$,

$$E_x = \{\omega = u\theta \colon u \in \mathbb{R}_+\}.$$

By (6), we have

$$d\omega = iuh_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + \omega \wedge \left(\frac{du}{u} + \phi_0\right),$$

which can be written

$$d\omega = ig_{\alpha\bar{\beta}}\omega^\alpha \wedge \omega^{\bar{\beta}} + \omega \wedge \phi, \tag{7}$$

where $g_{\alpha\bar{\beta}}$ is a constant matrix and $(\omega, \omega^\alpha, \omega^{\bar{\beta}}, \phi)$ is a coframe on $E$.

# The bundle of CTCM coframes $Y \to E \to M$.

The coframe $(\omega, \omega^\alpha, \omega^{\bar\beta}, \phi)$ on $E$ is determined up to

$$\begin{pmatrix} \tilde\omega \\ \tilde\omega^\alpha \\ \tilde\omega^{\bar\beta} \\ \tilde\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v^\alpha & {v_\nu}^\alpha & 0 & 0 \\ v^{\bar\beta} & 0 & {v_{\bar\mu}}^{\bar\beta} & 0 \\ s & ig_{\gamma\bar\rho}{v_\nu}^\gamma v^{\bar\rho} & -ig_{\gamma\bar\rho}{v_{\bar\mu}}^{\bar\rho} v^\gamma & 1 \end{pmatrix} \begin{pmatrix} \omega \\ \omega^\nu \\ \omega^{\bar\mu} \\ \phi \end{pmatrix}, \tag{8}$$

where $g_{\alpha\bar\beta} = g_{\nu\bar\mu}{v_\alpha}^\nu {v_{\bar\beta}}^{\bar\mu}$.

- Let $Y \to E$ be the bundle of all CTCM coframes, i.e., the bundle of all coframes of the form (8).
- $G$ = group of all matrices in (8) acts on $Y \to E$.

# Reduction of $Y \to E \to M$ to a $\{e\}$-structure [3].

## Theorem CM-2

There exists a uniquely determined coframe

$$(\omega, \omega^\alpha, \omega^{\bar\beta}, \phi, \phi^\alpha, \phi^{\bar\beta}, \phi_\gamma{}^\alpha, \phi_{\bar\mu}{}^{\bar\beta}, \psi) \tag{9}$$

on $Y$ that satisfy **structure equations** (including (7)). The coframe can be assembled into a Cartan connection on $Y \to E \to M$.

- One of the structure equations has the form

$$d\phi_\gamma{}^\alpha = \phi_\gamma{}^\nu \wedge \phi_\nu{}^\alpha + S_\gamma{}^\alpha{}_{\nu\bar\mu} \omega^\nu \wedge \omega^{\bar\mu} + \ldots. \tag{10}$$

- Given a CTCM coframe $(\theta, \theta^\alpha, \theta^{\bar\beta}, \phi)$ ($\implies$ section of $Y$), the forms (9) can be pulled back to $M$, and $S_{\alpha\bar\beta\nu\bar\mu} = g_{\gamma\bar\beta} S_\gamma{}^\alpha{}_{\nu\bar\mu}$ yields the CR curvature tensor on $M$ previously defined.

# Locally "spherical" CR structures.

**Remark.** Bianchi identities can be used to show that if

$$S_{\alpha\bar{\beta}\nu\bar{\mu}} \equiv 0, \quad \text{on } \pi^{-1}(U) \subset Y$$

for some $U \subset M$, then the coframe (9) on $\pi\colon Y \to M$ (locally over $U$) coincides with (satisfies the same structure equations as) that of the hyperquadric $\pi\colon Y_0 \to Q^n_\ell$. According to E. Cartan's solution to his "equivalence problem", it follows that there is a diffeomorphism $Y \cong Y_0$ (locally). This pushes down to a CR equivalence $U \cong U' \subset Q^n_\ell$.

- Thus, $S_{\alpha\bar{\beta}\nu\bar{\mu}} \equiv 0$ characterizes the hyperquadric locally.

S. Webster and N. Tanaka's approach. Pseudohermitian geometry.

# Pseudohermitian geometry and admissible frames

**Fix** a contact form $\theta$ on $M$. $(M, \mathcal{V} = T^{0,1}M, \theta)$ is called a pseudohermitian manifold. Let $(\theta, \theta^\alpha)$ be a CR coframe. By a change

$$\theta^\alpha \mapsto \theta^\alpha + u^\alpha \theta,$$

it follows from (6) that we can achieve

$$d\theta = i h_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}. \tag{11}$$

A CR coframe $(\theta, \theta^\alpha)$ satisfying (11) is called admissible. The forms $\theta^\alpha$ are determined up to changes

$$\theta^\alpha \mapsto {u_\nu}^\alpha \theta^\nu, \quad h_{\alpha\bar{\beta}} = h_{\nu\bar{\mu}} {u_\alpha}^\nu {u_{\bar{\beta}}}^{\bar{\mu}}.$$

# The pseudohermitian connection [6, 5].

## Theorem ΨH

Given an admissible CR coframe $(\theta, \theta^\alpha)$, there are uniquely determined connection forms ${\omega_\nu}^\beta$, torsion forms $\tau^\alpha = {A^\alpha}_{\bar\mu}\theta^{\bar\mu}$ such that

$$\begin{aligned} d\theta &= ih_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta} \\ d\theta^\alpha &= \theta^\nu \wedge {\omega_\nu}^\alpha + \theta \wedge \tau^\alpha \\ dh_{\alpha\bar\beta} &= \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha}. \end{aligned} \tag{12}$$

The connection forms satisfy

$$d{\omega_\alpha}^\beta = {\omega_\alpha}^\nu \wedge {\omega_\nu}^\beta + {R_\alpha}^\beta{}_{\nu\bar\mu}\theta^\nu \wedge \theta^{\bar\mu} + \dots. \tag{13}$$

(+ similar equation for the torsion forms.)

# Remarks.

- $R_{\alpha\bar{\beta}\nu\bar{\mu}}$ is called the Tanaka-Webster curvature. Pseudohermitian (but not a CR) invariant.
- $M$ is torsion free (i.e. $\tau^\alpha = 0$) $\iff$ the Reeb vector field is an infinitesimal CR automorphism.
- On a CR manifold $M$, there is a pseudohermitian structure that is torsion free $\iff$ there is a transverse infinitesimal CR automorphism; such $M$ are called "rigid" or "regular".
- The CR structure on $M$ is in some sense the "conformal class" of pseudohermitian structures on $M$.

# Tanaka-Webster curvature vs. CR curvature

Fix a pseudohermitian structure $\theta$, and let $(\theta, \theta^\alpha)$ be an admissible coframe. Then $(\theta, \theta^\alpha, \theta^{\bar\beta}, \phi = 0)$ is a CTCM coframe. We pull down the CR curvature $S_{\alpha\bar\beta\nu\bar\mu}$ using this CTCM coframe.

## Proposition

The CR curvature is the traceless part (Weyl tensor) of the Tanaka-Webster curvature; i.e.,

$$S_{\alpha\bar\beta\mu\bar\nu} = R_{\alpha\bar\beta\mu\bar\nu} - \frac{R_{\alpha\bar\beta}h_{\mu\bar\nu} + R_{\mu\bar\beta}h_{\alpha\bar\nu} + R_{\alpha\bar\nu}h_{\mu\bar\beta} + R_{\mu\bar\nu}h_{\alpha\bar\beta}}{n+2} + \frac{R(h_{\alpha\bar\beta}h_{\mu\bar\nu} + h_{\alpha\bar\nu}h_{\mu\bar\beta})}{(n+1)(n+2)},$$

where

$$R_{\alpha\bar\beta} := R_\mu{}^\mu{}_{\alpha\bar\beta} \text{ and } R := R_\mu{}^\mu$$

are respectively the *pseudohermitian Ricci* and *scalar curvatures* of $(M, \theta)$.

C. Fefferman's approach. Just kidding!

The End

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