CR Geometry, Mappings into Spheres, and Sums-Of-Squares
Lecture VI

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Outline - Lecture VI

The unit sphere $\mathbb{S}^n \subset \mathbb{C}^{n+1}$.

- Let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{C}^{n+1}$.
- $\mathbb{S}^n$ is a compact strictly pseudoconvex CR manifold of CR dim n. (Superscript n refers to its CR dimension. The dimension is $2n+1$.)

$$\sum_{j-1}^{n+1} |z_j|^2 - 1 = 0; \quad \|z\|^2 - 1 = 0.$$

- Notation: For $a = (a_1, \ldots, a_m)$,

$$\|a\|^2 := \sum_{j=1}^{m} |a_j|^2.$$

- $\mathbb{S}^n$ is locally the flat model for strictly pseudoconvex CR manifolds M^n.
- Understanding CR submanifold structure of $\mathbb{S}^n$ is a fundamental problem. Start with locally spherical submanifolds.

Automorphisms of $\mathbb{S}^N$. Spherical equivalence of CR maps.

- $\mathrm{Aut}(\mathbb{S}^N) \subset \mathrm{Aut}(\mathbb{CP}^N)$; $T \in \mathrm{Aut}(\mathbb{S}^N) \iff T$ rational map of degree one that preserves $\mathbb{S}^N$; such T have no poles in the closed unit ball.
- $T \in \mathrm{Aut}(\mathbb{S}^N)$ is called a "rigid motion" in $\mathbb{S}^N$.
- If $\Sigma \subset \mathbb{S}^N$ is a submanifold, then we consider $T(\Sigma) \subset \mathbb{S}^N$, for $T \in \mathrm{Aut}(\mathbb{S}^N)$, to be equivalent; $T(\Sigma)$ and $S(\Sigma)$ are spherically equivalent if $S, T \in \mathrm{Aut}(\mathbb{S}^N)$.
- If $f_1, f_2 \colon M^n \to \mathbb{S}^N$ are CR maps, then f_1, f_2 are spherically equivalent if $f_2 = T \circ f_1 \circ \phi \colon M^n \to \mathbb{S}^N$ for $T \in \mathrm{Aut}(\mathbb{S}^N)$, $\phi \in \mathrm{Aut}(M^n)$. **Notation:** $f_1 \sim f_2$.
- Important special case: $M^n = \mathbb{S}^n$; or $M^n \subset \mathbb{S}^n$.

Poincaré-Alexander Theorem.

Poincaré-Alexander Theorem. ([17, 1])

Let $M = M^n \subset \mathbb{S}^n$ be open, and $f \colon M^n \to \mathbb{S}^n$ smooth, nonconstant CR map. Then $f = T|_M$ for some $T \in \mathrm{Aut}(\mathbb{S}^n)$.

Theorem. (Forstneric, Cima-Suffridge[11, 3])

Let $M = M^n \subset \mathbb{S}^n$ be open, and $f \colon M^n \to \mathbb{S}^N$ smooth, nonconstant CR map. Then $f = R|_M$ for some rational map R without poles on the closed unit ball.

Theorem. (Faran, Cima-Suffridge [10, 2])

Let $M = M^n \subset \mathbb{S}^n$ be open, and $f \colon M^n \to \mathbb{S}^N$ smooth, nonconstant CR map. Assume $N - n < n$. Then $f = (T \circ L)|_M$, where $T \in \mathrm{Aut}(\mathbb{S}^N)$ and $L \colon \mathbb{C}^{n+1} \to \mathbb{C}^{N+1}$ the standard linear embedding; $L(z) = (z, 0)$.

CR maps $f : \mathbb{S}^n \to \mathbb{S}^N$ in higher codimension $N - n$.

- $N - n < n$. Rigidity: $f = T \circ L$ (Webster, Faran, Huang, ...).
- $N - n = n$. Two maps: $f = T \circ L$ or $f = T \circ W$ (Huang-Ji [14])

$$W(z_1, \ldots, z_{n+1}) := (z_1, \ldots, z_n, z_1 z_{n+1}, z_2 z_{n+1}, \ldots, z_{n+1}^2).$$

- $N - n = n + 1$. 1-parameter family (D'Angelo [4], Hamada [12]):

$$D_t(z_1, \ldots, z_{n+1}) := (z_1, \ldots, z_n, s_t z_{n+1}, c_t z_1 z_{n+1}, \ldots, c_t z_{n+1}^2), \quad (1)$$

  where $c_t = \cos t$, $s_t = \sin t$, $t \in [0, \pi/2]$.
- $N - n < 2n - 1$. Nothing new! $f = T \circ L \circ D_t$ (Huang-Ji-Xu [15]).
- $2n - 1 \leq N - n \leq 2n + 2$. New maps: "generalized" Whitney maps + families.
- $N - n < 3n - 3$. Nothing new! $f = T \circ L \circ f_0$ (Huang-Ji-Yin [16]).

Minimally embedded CR maps $f: M^n \to \mathbb{S}^N$.

Definition

- $f: M^n \to \mathbb{S}^N$ is minimally embedded if $f(M^n)$ is not contained in a proper complex plane (affine subspace) section of $\mathbb{S}^N$, i.e. f does not "come from" a lower codimensional mapping $f_0: M^n \to \mathbb{S}^{N_0}$ with $N_0 < N$.
- If $f(M^n)$ is contained in a proper complex plane section of dimension $N_0 + 1$, then we say that f is N_0-flat. I.e., "$f(M^n) \subset \mathbb{S}^{N_0}$".

The HJY Gap Conjecture; $f : \mathbb{S}^n \to \mathbb{S}^N$.

- $0 \leq N - n < n$: All maps are n-flat (1st Gap; Faran '86, ...).
- $n + 1 < N - n < n + (n-1) = 2n - 1$: All maps are $2n+1$-flat (2nd Gap; Huang-Ji-X0 '06).
- $2n + 2 < N - n < n + (n-1) + (n-2) = 3n - 3$: All maps are $3n+2$-flat (3rd Gap; Huang-Ji-Yin '12).

  $\vdots$
- $(k-1)n + k - 1 < N - n < n + (n-1) + \ldots + (n-k+1)$: All maps are $kn + k - 1$-flat (kth Gap; conjectural for $k \geq 4$).

  $\vdots$
- After that ($k = k_0 \sim \sqrt{2n}$): No more gaps (eventually D'Angelo-Han-Lebl '07 [5]).

A Sums-Of-Squares (SOS) Problem.

- Let $A(z, \bar{z})$ be a Hermitian (real) polynomial in $(z, \bar{z})$, with $z \in \mathbb{C}^n$.
- Linear algebra $\implies A(z, \bar{z}) = \|p(z)\|^2 - \|q(z)\|^2$, where $p = (p_1, \ldots, p_s)$, $q = (q_1, \ldots, q_t)$ are linearly independent polynomials in z.
- The rank of A is $\operatorname{rank} A = s + t$.
- A is an SOS if $A(z, \bar{z}) = \|p(z)\|^2$.

An SOS Problem. Assume $A(z, \bar{z})\|z\|^2$ is an SOS. What are possible ranks of $A(z, \bar{z})\|z\|^2$?

Huang's Lemma ([13])

If $A(z, \bar{z})\|z\|^2$ is an SOS, then $\operatorname{rank} A\|z\|^2 = 0$ or $\geq n$.

The SOS Conjecture.

The Gap Conjecture follows from the following SOS Conjecture, in view of a result of the speaker ([7]).

SOS Conjecture

For any Hermitian polynomial $A(z,\bar{z})$ such that $\|z\|^2 A(z,\bar{z})$ is an SOS, the rank r of $\|z\|^2 A(z,\bar{z})$ can only take the values:

$$
\begin{aligned}
&0, \\
&n, \\
&2n-1, 2n, \\
&\vdots \\
&n+(n-1)+\ldots+(n-k+1), \ldots, kn, \\
&\text{or} \\
&\geq n+(n-1)+\ldots+(n-k_0+1) = k_0 n - k_0(k_0-1)/2
\end{aligned}
\tag{2}
$$

Second Fundamental Form and Covariant Derivatives

For $M = M^n$ and $\widehat{M} = \widehat{M}^N$ strictly ψcvx, $f\colon M \to \widehat{M}$ transversal CR immersion, $p \in M$,

$$\mathrm{SFF}_p\colon\ T_p^{1,0}M \times T_p^{1,0}M \to T_{f(p)}^{1,0}\widehat{M}/f_*(T_p^{1,0}M),$$

and, for $m \in \mathbb{N}$,

$$\nabla^m\mathrm{SFF}_p\colon\ T_p^{1,0}M \times \ldots \times T_p^{1,0}M \to T_{f(p)}^{1,0}\widehat{M}/f_*(T_p^{1,0}M).$$

Given local frames $L_1, \ldots, L_n$, and $\hat{L}_1, \ldots, \hat{L}_{N-n}$ for $T^{1,0}M \cong \mathbb{C}^n$ and $T^{1,0}\widehat{M}/f_*(T^{1,0}M) \cong \mathbb{C}^{N-n}$,

$$\mathrm{SFF}_p \sim {\omega_\alpha}^a{}_\beta, \quad \nabla^m\mathrm{SFF}_p \sim {\omega_\alpha}^a{}_{\beta;\gamma_1\ldots\gamma_m},$$

where $\alpha, \beta, \gamma_j \in \{1, \ldots, n\}$ and $a \in \{1, \ldots, N-n\}$.

Span of SFF; Spaces $E_m(p)$

Introduce:

$$E_2(p) \subset E_3(p) \subset \ldots \subset E_m(p) \subset \ldots \subset \mathbb{C}^{N-n},$$

by

$$E_m(p) := \operatorname{span}_{\mathbb{C}} \left\langle (\omega_{\alpha_1}{}^{a}{}_{\alpha_2;\alpha_3\ldots\alpha_j})_{a=1}^{N-n} \colon \alpha_1, \ldots, \alpha_j \in \{1, \ldots, n\},\ j \leq m \right\rangle.$$

Degeneracy of maps

Definition

- $f\colon M^n \to \widehat{M}^N$ is (m,d)-nondegenerate at $p \in M$ if $\dim E_m(p) = d$, where $E_{m-1}(p) \subsetneq E_m(p) = E_{m+1}(p) = \ldots$.
- Set $d_l(p) := \dim E_l(p)$, $l = 2, 3, \ldots, m$; for convenience, also set $d_1(p) = 0$. Note: $d = d_m(p)$.

Note:

- $0 = d_1(p) \leq d_2(p) < \ldots < d_{m-1}(p) < d_m(p) = d(p) \leq N - n$.
- At generic p, $d_l(p)$ are locally constant $= d_l$.

Flat Maps; $f\colon M^n \to S^N \subset \mathbb{C}^{N+1}$

Observation 1:

If $f(M^n)$ is contained in a complex plane $P = P^{N_0+1} \subset \mathbb{C}^{N+1}$, then $d(p) \le N_0 - n$ for all $p \in M$.

Converse? Well, if $d(p)$ is "small":

Theorem ([8, 9])

If $d(p) \le d$ for $p \in M$ and $d < n$, then $f(M^n)$ is contained in a complex plane $P = P^{N_0+1}$ with $N_0 - n = d$ (i.e. $N_0 = n + d$).

And if $d(p)$ is larger?

"Observation" 2

Let $f = D_t\colon S^n \to S^N$, with D_t given by (1) and $N - n = n + 1$. Then, $d(p) \le n$ for $p \in S^n$, but $f(S^n)$ is minimally embedded (not contained in a plane $P = P^{2n+1} \subset \mathbb{C}^{2n+2}$), except for $t = 0$ (Whitney) and $t = \pi/2$ (linear).

The d_l measure flatness.

Theorem 1 (E. '13 [7])

Let $f\colon M^n \to S^N$, and assume that there are integers

$$0 \leq k_2, k_3, \ldots, k_m \leq n-1$$

($m =$ degeneracy of f) such that, for $p \in M$:

$$d_l(p) - d_{l-1}(p) < \sum_{i=0}^{k_l}(n-i), \quad l = 2, \ldots, m, \quad (d_1(p) = 0).$$

If $k < n$, where

$$k := \sum_{l=2}^{m} k_l, \quad d := \max_{p \in M} d(p)$$

then $f(M)$ is contained in a complex plane P^{N_0+1} with $N_0 - n = d + k$.

Total second fundamental polynomial of $f: M^n \to \mathbb{S}^N$.

- For $l \geq 2$, set $\Omega_{(l)} = (\Omega^1_{(l)}, \dots, \Omega^{N-n}_{(l)})$, where

$$\Omega^j_{(l)}(z) := \omega_{\gamma_1}{}^{a}{}_{\gamma_2;\gamma_3\dots\gamma_l} z^{\gamma_1} \dots z^{\gamma_l}, \quad a = n + j, \tag{3}$$

- The total second fundamental polynomial $\Omega = (\Omega^1, \dots, \Omega^{N-n})$ of f is

$$\Omega^j(z) := \sum_{l=2}^{m} \Omega^j_{(l)}(z), \tag{4}$$

where m is the integer where the dimensions d_l stabilize.

- Linear algebra $\implies$ $d = \operatorname{rank} \|\Omega(z)\|^2$.

Total Polynomial Gauss Equation for $f: \mathbb{S}^n \to \mathbb{S}^N$ ([6])

There exists a Hermitian polynomial $A(z, \bar{z})$ such that

$$\|\Omega(z)\|^2 = A(z, \bar{z})\|z\|^2. \tag{5}$$

The SOS Conjecture $\implies$ the Gap Conjecture.

Sketch of proof ([6]). Let $f\colon \mathbb{S}^n \to \mathbb{S}^N$ and assume that $N-n$ is in pth gap

$$(p-1)n+p-1 < N-n < n+(n-1)+\ldots+(n-p+1), \quad p \le k_0.$$

Gap Conjecture predicts that f is $(pn+p-1)$-flat. The rank of nondegeneracy $d \le N-n < n+(n-1)+\ldots+(n-p+1)$. Total Polynomial Gauss Equation and SOS Conjecture $\implies d \le (p-1)n$.

Proposition

There exist $k_2,\ldots,k_m$, $k=k_2+\ldots+k_m$ as in Theorem 1 such that $k \le p-1$.

Theorem 1 $\implies f$ is $(n+d+k)$-flat; and

$$n+d+k \le n+(p-1)n+(p-1) = pn+p-1.$$

□

Thank You for Your Attention!

H. Alexander.
Holomorphic mappings from the ball and polydisc.
Math. Ann., 209:249–256, 1974.

J. A. Cima and T. J. Suffridge.
A reflection principle with applications to proper holomorphic mappings.
Math. Ann., 265(4):489–500, 1983.

J. A. Cima and T. J. Suffridge.
Boundary behavior of rational proper maps.
Duke Math. J., 60(1):135–138, 1990.

John P. D'Angelo.
Proper holomorphic maps between balls of different dimensions.
Michigan Math. J., 35(1):83–90, 1988.

John P. D'Angelo, Jiří Lebl, and Han Peters.
Degree estimates for polynomials constant on a hyperplane.
Michigan Math. J., 55(3):693–713, 2007.

Peter Ebenfelt.
On the HJY gap conjecture in CR geometry vs. the SOS conjecture for polynomials.
submitted; http://front.math.ucdavis.edu/1508.04205.

Peter Ebenfelt.
Partial rigidity of degenerate CR embeddings into spheres.
Adv. Math., 239:72–96, 2013.

Peter Ebenfelt, Xiaojun Huang, and Dmitry Zaitsev.
Rigidity of CR-immersions into spheres.
Comm. Anal. Geom., 12(3):631–670, 2004.

Peter Ebenfelt and Ravi Shroff.
Partial rigidity of CR embeddings of real hypersurfaces into hyperquadrics with small signature difference.
Comm. Anal. Geom., 23(1):159–190, 2015.

James J. Faran.
The linearity of proper holomorphic maps between balls in the low codimension case.

J. Differential Geom., 24(1):15–17, 1986.

Franc Forstnerič.
Proper holomorphic maps from balls.
Duke Math. J., 53(2):427–441, 1986.

Hidetaka Hamada.
Rational proper holomorphic maps from $\mathbf{B}^n$ into $\mathbf{B}^{2n}$.
Math. Ann., 331(3):693–711, 2005.

Xiaojun Huang.
On a linearity problem for proper holomorphic maps between balls in complex spaces of different dimensions.
J. Differential Geom., 51:13–33, 1999.

Xiaojun Huang and Shanyu Ji.
Mapping $\mathbb{B}^n$ into $\mathbb{B}^{2n-1}$.
Inventiones Mathematicae, 145:219–250, 2001.
10.1007/s002220100140.

Xiaojun Huang, Shanyu Ji, and Dekang Xu.

A new gap phenomenon for proper holomorphic mappings from B^n into B^N.
Math. Res. Lett., 13(4):515–529, 2006.

Xiaojun Huang, Shanyu Ji, and Wanke Yin.
On the third gap for proper holomorphic maps between balls.
Math. Ann., 358(1-2):115–142, 2014.

H. Poincaré.
Les fonctions analytiques de deux variables et la représentation conforme.
Rend. Circ. Mat. Palermo, 23(2):185–220, 1907.