Minimal Surfaces and Complex Analysis

Lecture 2: Mergelyan approximation and applications

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Lecture 2: Mergelyan approximation and applications

In this lecture we will

- discuss the Mergelyan approximation theorem for conformal minimal immersions $\mathbf{M} \to \mathbb{R}^n$ $(n \geq 3)$,
- discuss the general position theorem for minimal surfaces in $\mathbb{R}^n$ $(n \geq 5)$,

and use them to construct proper minimal immersions $\mathbf{M} \to \mathbb{R}^n$, embeddings if $n \geq 5$.

Based on joint work with

- **Franc Forstnerič**, University of Ljubljana.
- **Francisco J. López**, University of Granada.

[A. Alarcón, F.J. López: *Minimal surfaces in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$.* J. Differential Geom. 2012]
[A. Alarcón, F. Forstnerič: *Null curves and directed immersions of open Riemann surfaces.* Invent. Math. 2014]
[A. Alarcón, F. Forstnerič, F.J. López: *Embedded minimal surfaces in $\mathbb{R}^n$.* Preprint 2014]

Mergelyan Theorem

A compact subset K of an open Riemann surface $\mathbf{M}$ is said to be **Runge** if $\mathbf{M} \setminus K$ has no relatively compact connected components.

1951 **Mergelyan** If $K \subset \mathbf{M}$ is a Runge compact subset then every continuous function $K \to \mathbb{C}$, holomorphic on $\overline{\mathring{K}}$, may be approximated, uniformly on K, by entire functions $\mathbf{M} \to \mathbb{C}$.

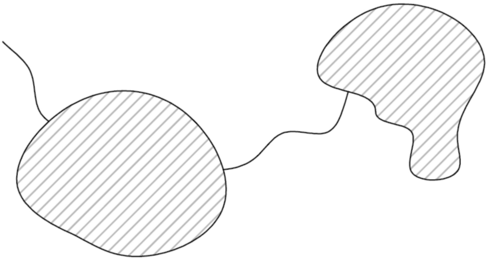
Mergelyan Theorem for conformal minimal immersions

Definition

A compact subset $\mathbf{S}$ of an open Riemann surface $\mathbf{M}$ is said **admissible** if

$$\mathbf{S} = K \cup \Gamma,$$

where $K = \bigcup \overline{D}_j$ is a union of finitely many pairwise disjoint, compact, smoothly bounded domains $\overline{D}_j$ in $\mathbf{M}$ and $\Gamma = \bigcup \Gamma_i$ is a union of finitely many pairwise disjoint smooth arcs or closed curves that intersect K only in their endpoints (or not at all), and such that their intersections with the boundary bK are transverse.



Mergelyan Theorem for conformal minimal immersions

Definition

If $\mathbf{S} = K \cup \Gamma \subset \mathbf{M}$ is admissible, a **generalized conformal minimal immersion** of $\mathbf{S}$ into $\mathbb{R}^n$ $(n \geq 3)$ is a pair $(\mathbf{X}, f\vartheta)$ where

- $\mathbf{X} \colon \mathbf{S} \to \mathbb{R}^n$ is a smooth immersion,
- $f \colon \mathbf{S} \to \mathfrak{A}^* = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^n z_j^2 = 0\} \setminus \{0\}$ is continuous, holomorphic on K,
- ϑ is a holomorphic 1-form on $\mathbf{M}$ with no zeros, and

$$\mathbf{X}(z) = \mathbf{X}(z_0) + \Re \int_{z_0}^{z} f\vartheta, \quad z_0, z \in S.$$

We denote by $\mathrm{GCMI}(\mathbf{S}, \mathbb{R}^n)$ the set of generalized conformal minimal immersions of $\mathbf{S} \subset \mathbf{M}$ into $\mathbb{R}^n$.

If $\mathbf{S} = K \cup \Gamma$ and $(\mathbf{X}, f\vartheta) \in \mathrm{GCMI}(\mathbf{S}, \mathbb{R}^n)$ then $\mathbf{X}|_K \colon K \to \mathbb{R}^n$ is a conformal minimal immersion.

Mergelyan Theorem for conformal minimal immersions

Given $A \subset \mathbf{M}$ and $n \geq 3$ we denote by $\mathrm{CMI}(A, \mathbb{R}^n)$ the set of conformal minimal immersions $A \to \mathbb{R}^n$.

Definition

An immersion $\mathbf{X} \in \mathrm{CMI}(A, \mathbb{R}^n)$ is said to be **nondegenerate** if $f(A) \subset \mathfrak{A}^*$ is not contained in any linear complex hyperplane of $\mathbb{C}^n$, where $f = \partial \mathbf{X}/\vartheta$.

We denote by $\mathrm{CMI}_*(A, \mathbb{R}^n)$ the set of nondegenerate conformal minimal immersions $A \to \mathbb{R}^n$.

Theorem

Assume that $\mathbf{M}$ *is an open Riemann surface and that* $\mathbf{S} = K \cup \Gamma$ *is a compact* **Runge** *admissible set in* $\mathbf{M}$. *Then every* **generalized conformal minimal immersion** $(\mathbf{X}, f\vartheta) \in \mathrm{GCMI}(\mathbf{S}, \mathbb{R}^n)$ $(n \geq 3)$ *may be approximated in the* $\mathscr{C}^1(\mathbf{S})$ *topology by nondegenerate conformal minimal immersions* $\mathbf{Y} \in \mathrm{CMI}_*(\mathbf{M}, \mathbb{R}^n)$.

Approach in $\mathbb{R}^3$: The López-Ros transformation

In dimension $n = 3$, the Weierstrass data $f\vartheta$ of a conformal minimal immersion $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^3$ are of the form

$$f = \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{\imath}{2}\left(\frac{1}{g} + g\right), 1 \right) \phi_3,$$

where g is meromorphic (the Complex Gauss map) and ϕ_3 holomorphic. If $\mathbf{M}$ is simply connected and $\widetilde{g} = gh$ where h is holomorphic and has no zeros on $\mathbf{M}$, then

$$\widetilde{f}\vartheta = \left(\frac{1}{2}\left(\frac{1}{\widetilde{g}} - \widetilde{g}\right), \frac{\imath}{2}\left(\frac{1}{\widetilde{g}} + \widetilde{g}\right), 1 \right) \phi_3\vartheta,$$

integrates to a conformal minimal immersion $\widetilde{\mathbf{X}}\colon \mathbf{M} \to \mathbb{R}^3$ by the formula

$$\widetilde{\mathbf{X}}(p) = \mathbf{X}(p_0) + \Re \int_{p_0}^{p} \widetilde{f}\vartheta.$$

If $\mathbf{M}$ is not simply connected, one has to deal with the period problem of the first two components; note that the third components $\widetilde{\mathbf{X}}_3 = \mathbf{X}_3$.
$\widetilde{\mathbf{X}}$ is said to be obtained from $\mathbf{X}$ by a **López-Ros deformation**.

Approach in $\mathbb{R}^3$: The López-Ros transformation

To prove the Mergelyan Theorem for CMIs into $\mathbb{R}^3$ we argue as follows:

Let $\mathbf{S} = K \cup \Gamma$ be a compact Runge admissible set in $\mathbf{M}$ and let $(\mathbf{X}, f\vartheta) \in \mathrm{GCMI}(\mathbf{S}, \mathbb{R}^3)$ be a generalized conformal minimal immersion.

Then f is of the form

$$f = \left(\frac{1}{2}\left(\frac{1}{g} - g\right) \, , \, \frac{\imath}{2}\left(\frac{1}{g} + g\right) \, , \, 1 \right) \phi_3,$$

where g and ϕ_3 are smooth on $\mathbf{S}$ (except for the poles of g which we may assume that lie on K), g is meromorphic on K, and ϕ_3 is holomorphic on K. By the classical Mergelyan Theorem we may approximate g and ϕ_3, uniformly on K, by a meromorphic function $\widetilde{g}$ and a holomorphic function $\widetilde{\phi}_3$ on $\mathbf{M}$ such that

$$\widetilde{f}\vartheta = \left(\frac{1}{2}\left(\frac{1}{\widetilde{g}} - \widetilde{g}\right) \, , \, \frac{\imath}{2}\left(\frac{1}{\widetilde{g}} + \widetilde{g}\right) \, , \, 1 \right) \widetilde{\phi}_3\vartheta$$

is holomorphic and has no real periods on $\mathbf{M}$. Integrating $\widetilde{f}\vartheta$ and taking real part solves the theorem.

Approach in $\mathbb{R}^3$: The López-Ros transformation

The above approach also works in the **nonorientable** framework (one has to take care of the symmetries

$$\phi_3 \circ \mathfrak{I} = \overline{\phi}_3, \quad g \circ \mathfrak{I} = -\frac{1}{\overline{g}},$$

where $\mathfrak{I} \colon \mathbf{M} \to \mathbf{M}$ is an antiholomorphic involution without fixed points and ϑ satisfies $\mathfrak{I}^* \vartheta = \overline{\vartheta}$)...

[A. Alarcón, F.J. López: *Approximation theory for non-orientable minimal surfaces and applications.* Geom. Topol. (2015)]

2014 López ... and also for minimal surface of **finite total curvature** (one has to look for algebraic g, ϕ_3, and ϑ)...

... but it does not work for $n \geq 4$.

Stein Manifolds

A complex manifold S is said to be a **Stein manifold** if:

- **holomorphic functions on S separate points**:

$$x, x' \in S,\ x \neq x' \Leftrightarrow f(x) \neq f(x') \text{ for some } f \in \mathcal{O}(S).$$

- **S is holomorphically convex**: For every compact set $K \subset S$, its $\mathcal{O}(S)$-convex hull

$$\widehat{K}_{\mathcal{O}(S)} := \{x \in S \colon |f(x)| \leq \sup_K |f| \ \forall f \in \mathcal{O}(S)\} \subset S$$

  is also compact.

  Equivalently, for every discrete sequence $\{a_j\}_{j \in \mathbb{N}} \subset S$ there exists a holomorphic function f on S such that $\{|f(a_j)|\}_{j \in \mathbb{N}} \to \infty$.

Every **open Riemann surface** is Stein.

What should be the dual notion to a Stein manifold?

By definition, a Stein manifold S admits **many** holomorphic functions $f\colon S \to \mathbb{C}$.

Replace $\mathbb{C}$ by a complex manifold Z and ask the following:

For which complex manifolds Z do there exist many holomorphic maps $S \to Z$ from any Stein manifold S?

Oka manifolds

What is a good way to interpret **many** maps?

Weierstrass Theorem. On a discrete subset of a domain $\Omega \subset \mathbb{C}$ we may prescribe the values of a holomorphic function $\Omega \to \mathbb{C}$.

Runge Theorem. If $K \subset \mathbb{C}$ is a compact set such that $\mathbb{C} \setminus K$ is connected, then every holomorphic function $K \to \mathbb{C}$ may be approximated uniformly on K by entire functions $\mathbb{C} \to \mathbb{C}$.

Cartan Extension Theorem. If T is a closed complex subvariety of a Stein manifold S, then every holomorphic function $T \to \mathbb{C}$ extends to a holomorphic function $S \to \mathbb{C}$.

Oka-Weil Approximation Theorem. Let $K = \widetilde{K}_{\mathcal{O}(S)}$ be a compact holomorphically convex subset of a Stein manifold S. Here

$$\widehat{K}_{\mathcal{O}(S)} := \{p \in S \colon |f(p)| \leq \sup_K |f| \ \forall f \in \mathcal{O}(S)\}.$$

Then every holomorphic function $K \to \mathbb{C}$ can be approximated uniformly on K by holomorphic functions $S \to \mathbb{C}$.

Oka manifolds

Definition

A complex manifold Z is said to be an **Oka manifold** if every holomorphic map from a neighborhood of a compact convex set $K \subset \mathbb{C}^N$ to Z may be approximated, uniformly on K, by entire maps $\mathbb{C}^N \to Z$.

Basic Oka Property with Approximation: Z is Oka if and only if maps $K \to Z$, where K is a holomorphically convex compact subset of a Stein manifold S, enjoy the Runge property.

1989 Gromov **Elliptic** complex manifolds are Oka.

A complex manifold Z is **elliptic** if it admits a **dominating spray**; i.e., a family of holomorphic maps $f_x \colon \mathbb{C}^n \to Z$, depending holomorphically on $x \in Z$, such that $f_x(0) = x$ and $df_x(0) \colon \mathbb{C}^n \to T_x Z$ is surjective for every $x \in Z$.

Example If Z admits $\mathbb{C}$-complete holomorphic vector fields $V_1, \ldots, V_n$ which span $T_x Z$ at every point $x \in Z$, then the composition of their flows is a dominating spray on Z:

$$f_x(t_1, \ldots, t_n) = \phi^1_{t_1} \circ \phi^2_{t_2} \circ \cdots \circ \phi^n_{t_n}(x), \quad t_1, \ldots, t_n \in \mathbb{C}.$$

The Null Quadric is Oka

Theorem

The Null quadric $\mathfrak{A}^* = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^n z_j^2 = 0\} \setminus 0 \subset \mathbb{C}^n$ *is an* ***elliptic manifold****, and hence an* ***Oka manifold****.*

Indeed, the holomorphic vector fields on $\mathbb{C}^n$,

$$V_{j,k}(z) = z_j \frac{\partial}{\partial z_k} - z_k \frac{\partial}{\partial z_j}, \quad 1 \le j, k \le n,$$

are linear and hence $\mathbb{C}$-complete, their flows preserve $\mathfrak{A}^*$, and they span the tangent space of $\mathfrak{A}^*$ at every point. Thus $\mathfrak{A}^*$ is elliptic, and hence an Oka manifold by Gromov's theorem.

Dominating holomorphic sprays of maps

Let $\mathbf{M} = \mathring{\mathbf{M}} \cup b\mathbf{M}$ be a compact bordered Riemann surface, and let Z be a complex manifold.
We denote by $\mathscr{A}^r(\mathbf{M}, Z)$ the space of $\mathscr{C}^r$ maps $\mathbf{M} \to Z$ which are holomorphic in $\mathring{\mathbf{M}}$.

Definition

- A **holomorphic spray of maps** of class $\mathscr{A}^r(\mathbf{M}, Z)$ is a family of maps $f_t \in \mathscr{A}^r(\mathbf{M}, Z)$ depending holomorphically on a parameter t in a ball $0 \in B \subset \mathbb{C}^N$. The map f_0 is the **core** of the spray.
- A spray f_t is **dominating** if $\partial_t|_{t=0} f_t(x) \colon \mathbb{C}^N \to T_{f_0(x)} Z$ is surjective for every $x \in \mathbf{M}$.

2007 Drinovec-Drnovšek, Forstnerič If $\mathbf{M}$ is a bordered Riemann surface and Z is any complex manifold, then every map $f_0 \in \mathscr{A}^r(\mathbf{M}, Z)$ $(r \in \mathbb{Z}_+)$ is the core map of a dominating holomorphic spray $f_t \in \mathscr{A}^r(\mathbf{M}, Z)$ $(t \in B \subset \mathbb{C}^N,\ N \text{ large})$.

The period map

Let $\mathbf{M} = \mathring{\mathbf{M}} \cup b\mathbf{M}$ be a compact bordered Riemann surface. Fix a nowhere vanishing holomorphic 1-form ϑ on $\mathbf{M}$.

Pick a basis $\{\gamma_j\}_{j=1}^{l}$ of the 1st homology group $H_1(\mathbf{M};\mathbb{Z})$. Let

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l) \colon \mathscr{A}^0(\mathbf{M}, \mathbb{C}^n) \to (\mathbb{C}^n)^l$$

be the **period map**. The j-th component, applied to $f \in \mathscr{A}^0(\mathbf{M}, \mathbb{C}^n)$, is

$$\mathcal{P}_j(f) = \int_{\gamma_j} f\vartheta = \int_0^1 f(\gamma_j(s))\vartheta(\gamma_j(s), \dot{\gamma}_j(s))\, ds \in \mathbb{C}^n.$$

By Stokes' theorem, the period map does not change under homotopic deformations of the loops $\gamma_j \colon [0,1] \to \mathbf{M}$.

The 1-form $f\vartheta$ **has no real periods** if its periods are imaginary:

$$\Re\big(\mathcal{P}_j(f)\big) = \Re \int_{\gamma_j} f\vartheta = 0, \quad j = 1, \ldots, l.$$

The 1-form $f\vartheta$ is **exact** if its periods vanish:

$$\mathcal{P}_j(f) = \int_{\gamma_j} f\vartheta = 0, \quad j = 1, \ldots, l.$$

Period dominating holomorphic sprays

Lemma

Let $\mathbf{M}$, ϑ, *and* $\gamma_1, \ldots, \gamma_l$ *be as above, and let* $n \in \mathbb{N}$, $n \geq 3$, *and* $r \in \mathbb{Z}_0$.

Assume that $f_0 \colon \mathbf{M} \to \mathfrak{A}^* \subset \mathbb{C}^n$ *is a map of class* $\mathscr{A}^r(\mathbf{M}, \mathfrak{A}^*)$ *which is* **nondegenerate**, *in the sense that* $f(\mathbf{M}) \subset \mathfrak{A}^*$ *is not contained in any linear complex hyperplane of* $\mathbb{C}^n$.

Then there exists a **dominating** *and* **period dominating** *spray* $f_t \colon \mathbf{M} \to \mathfrak{A}^*$ *of class* $\mathscr{A}^r(\mathbf{M})$ $(t \in B \subset \mathbb{C}^N$, N *large):*

$$\frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{P}(f_t) \colon \mathbb{C}^N \to (\mathbb{C}^n)^l \quad \text{is surjective.}$$

Furthermore, every degenerate map $f_0 \in \mathscr{A}^r(\mathbf{M}, \mathfrak{A}^*)$ *may be approximated arbitrarily closely by nondegenerate maps in* $\mathscr{A}^r(\mathbf{M}, \mathfrak{A}^*)$.

Construction of period dominating holomorphic sprays

We may assume that $\mathbf{M} \Subset \widetilde{\mathbf{M}}$ is a smoothly bounded domain in an open Riemann surface $\widetilde{\mathbf{M}}$. Let $C_j = \gamma_j([0,1]) \subset \mathbf{M}$ $(j = 1, \ldots, l)$ be a basis of $H_1(\mathbf{M}; \mathbb{Z})$ such that $C = \cup_j C_j \subset \mathbf{M}$ is Runge.

- The graph $G = \{(x, f_0(x)) \colon x \in \mathbf{M}\}$ has an open Stein neighborhood $\Omega \subset \widetilde{\mathbf{M}} \times \mathfrak{A}^*$.
- Choose holomorphic vector fields $V_1, \ldots, V_m$ on Ω which are tangential to $\mathfrak{A}^*$ and span the tangent space at each point.
- Consider the spray f_t given by

$$f_t(x) = \phi^1_{h_1(x)t_1} \circ \cdots \circ \phi^N_{h_N(x)t_N}(x), \quad t = (t_1, \ldots, t_N),$$

  where each ϕ^j is the flow of one of the vector fields V_k (possibly with repetitions) and h_j are holomorphic functions on $\mathbf{M}$.
- We have

$$\frac{\partial}{\partial t_j}\bigg|_{t=0} f_t(x) = h_j(x) V_j(f_0(x)).$$

  Choose the values of h_j in a suitable way on $C = \cup_j C_j$ and apply Mergelyan's approximation theorem to ensure that f_t is dominating and period dominating.

Banach manifold structure

Theorem

Let $\mathbf{M}$ *be a compact bordered Riemann surface and* $r \in \mathbb{Z}_+$*. The set*

$$\{f \in \mathscr{A}^r(\mathbf{M}, \mathfrak{A}^*) \colon f \text{ nondegenerate}, \Re(\mathcal{P}(f)) = 0\},$$

where $\mathfrak{A}^* \in \mathbb{C}^n$*, is a real analytic Banach manifold.*

Indeed, the set $\mathscr{A}^r(\mathbf{M}, Z)$ is a complex Banach manifold for any $r \in \mathbb{Z}_+$ and any complex manifold Z.

The existence of period dominating sprays shows that the equation

$$\Re(\mathcal{P}(f)) = 0 \quad (f \in \mathscr{A}^r(\mathbf{M}, \mathfrak{A}^*))$$

is of maximal rank if f is nondegenerate.

Corollary

Let $\mathbf{M}$ *be a compact bordered Riemann surface and* $n \geq 3$*,* $r \in \mathbb{N}$*. The space* $\mathrm{CMI}^r_*(\mathbf{M}, \mathbb{R}^n)$ *of nondegenerate conformal minimal immersions is a real analytic Banach manifold with the natural* $\mathscr{C}^r(\mathbf{M})$ *norm.*

Mergelyan Theorem for conformal minimal immersions

Theorem

Assume that $\mathbf{M}$ *is an open Riemann surface and that* $\mathbf{S} = K \cup \Gamma$ *is a compact* **Runge** *admissible set in* $\mathbf{M}$.

Then every **generalized conformal minimal immersion** $(\mathbf{X}, f\vartheta) \in \mathrm{GCMI}(\mathbf{S}, \mathbb{R}^n)$ $(n \geq 3)$ *may be approximated in the* $\mathscr{C}^1(\mathbf{S})$ *topology by nondegenerate conformal minimal immersions* $\mathbf{Y} \in \mathrm{CMI}_*(\mathbf{M}, \mathbb{R}^n)$.

Furthermore, the approximating immersion $\mathbf{Y}$ *may be chosen so that* $\mathrm{Flux}_{\mathbf{Y}} = \mathfrak{p}$ *where* $\mathfrak{p}\colon H_1(\mathbf{M}, \mathbb{Z}) \to \mathbb{R}^n$ *is any group homomorphism satisfying* $\mathfrak{p}(C) = \Im \int_C f\vartheta$ *for all closed curve* $C \subset \mathbf{S}$.

Mergelyan Theorem for CMIs - Proof

Choose an exhaustion $\mathbf{S} \Subset \mathbf{M}_1 \Subset \mathbf{M}_2 \Subset \cdots \Subset \cup_{j=1}^{\infty} \mathbf{M}_j = \mathbf{M}$ such that every $\mathbf{M}_j$ is a smooth Runge domain in $\mathbf{M}$ and $\mathbf{M}_{j+1}$ is obtained from $\mathbf{M}_j$ in one of the following ways:

(a) Noncritical case: $\mathbf{M}_j$ is a strong deformation retract of $\mathbf{M}_{j+1}$.

(b) Critical case, index 0: A new connected component.

(c) Critical case, index 1: $\mathbf{M}_{j+1} = \mathbf{M}_j \cup (\text{1-handle})$.

Call $\mathbf{M}_0 = \mathbf{S}$ and $f_0 = f$. We inductively construct a sequence of nondegenerate holomorphic maps $f_j \colon \mathbf{M}_j \to \mathfrak{A}_*$ such that

$$\int_C f_j \vartheta = \imath \mathfrak{p}(C) \quad \text{for all closed loop } C \subset \mathbf{M}_j.$$

and $f_{j+1}|_{\mathbf{M}_j}$ approximates f_j as close as desired.
Integrating the limit map $h := \lim_{j\to\infty} f_j$ will solve the problem:

$$\mathbf{Y}(p) = \mathbf{X}(p_0) + \Re \int_{p_0}^{p} h\vartheta, \quad p_0 \in \mathbf{S},\ p \in \mathbf{M}.$$

Mergelyan Theorem - Proof - Basis of the induction

Assume that $\mathbf{S} = \mathbf{M}_0 = K \cup \Gamma$ is connected and a strong deformation retract of $\mathbf{M}_1$. Recall that f is holomorphic on an open neighborhood $U \subset \mathbf{M}$ of K and is smooth on Γ. Assume also, by the above Lemma, that $f_0 = f$ is nondegenerate.

- Up to a shrinking of U around K we may find a period dominating spray of smooth maps $h_w \colon U \cup \Gamma \to \mathfrak{A}_*$ which are holomorphic on U and depend holomorphically on a parameter w in a ball $B \subset \mathbb{C}^N$, with $h_0 = f_0$.
- By Mergelyan approximation, we get a new holomorphic spray of maps $\widetilde{h}_w \colon V \supset \mathbf{S} \to \mathfrak{A}^*$ which approximates the initial spray h_w uniformly on $\mathbf{M}_0$, and uniformly with respect to the parameter w.
- Since $\mathfrak{A}^*$ is an Oka manifold, $\mathbf{M} \times B$ is a Stein manifold, and $\mathbf{S} = \mathbf{M}_0$ is Runge in $\mathbf{M}$ and a deformation retract of $\mathbf{M}_1$, we may approximate the spray $\widetilde{h}_w$ uniformly on $\mathbf{S}$ (and uniformly with respect to the parameter w) by a holomorphic spray of maps $g_w \colon \mathbf{M}_1 \to \mathfrak{A}^*$ for w in a slightly smaller ball $B' \subset B$.
- If both approximations made above are close enough then there exists $w_0 \in B'$ close to the origin such that the map $f_1 := g_{w_0} \colon \mathbf{M}_1 \to \mathfrak{A}^*$ satisfies the required conditions, by the period dominating condition.

Mergelyan Theorem - Proof - Inductive step

Noncritical case: Since $\mathbf{M}_j$ is a strong deformation retract of $\mathbf{M}_{j+1}$ the proof of the basis of the induction applies.

Critical case, Index 0: We reduce the proof to to noncritical case by defining f_{j+1} in the new component as any holomorphic map into $\mathfrak{A}^*$.

Critical case, Index 1: Now $\mathbf{M}_{j+1}$ retracts onto the union of $\mathbf{M}_j$ and a smooth arc $E_j \cong [0,1] \subset \mathbf{M}_{j+1}$ whose endpoints a_j, b_j belong to $\mathbf{M}_j$ and which is otherwise disjoint from $\mathbf{M}_j$, and $\mathbf{M}_j \cup E_j$ is Runge in $\mathbf{M}_{j+1}$. Further, we may assume that $\mathbf{M}_j \cup E_j$ is admissible.

- If a_j, b_j belong to different components of $\mathbf{M}_j$ then no new nontrivial loop appears in the 1st homology group and we proceed as in the basis of the induction.
- If a_j, b_j belong to the same component of $\mathbf{M}_j$ then E_j closes to a loop $\gamma \subset \mathbf{M}_j \cup E_j \subset \mathbf{M}_{j+1}$. Since f_j is nondegenerate, the convex hull of the algebraic variety $\mathfrak{A}^*$ equals $\mathbb{C}^n$, and $\int_\gamma f_j \vartheta = \imath \mathfrak{p}(\gamma)$, we may extend f_j to a smooth map $\widehat{f}_j \colon \mathbf{M}_j \cup E_j \to \mathfrak{A}^*$ such that
$$\int_\gamma \widehat{f}_j \vartheta = \imath \mathfrak{p}(\gamma).$$
We then proceed as in the basis of the induction.

General Position Theorem

Self-intersections of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ are stable under deformations.

Theorem

Let $\mathbf{M}$ *be an open Riemann surface.*

Every conformal minimal immersion $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n$ $(n \geq 5)$ *can be approximated uniformly on compacts in* $\mathbf{M}$ *by conformal minimal* **embeddings** $\mathbf{Y}\colon \mathbf{M} \to \mathbb{R}^n$.

Furthermore $\mathbf{Y}$ *may be chosen such that* $\mathrm{Flux}_{\mathbf{Y}} = \mathrm{Flux}_{\mathbf{X}}$.

General Position Theorem - Proof

It suffices to consider the case when $\mathbf{M}$ is a compact bordered Riemann surface. Consider the difference map

$$\delta\mathbf{X}\colon \mathbf{M}\times\mathbf{M}\to\mathbb{R}^n, \quad \delta\mathbf{X}(p,q)=\mathbf{X}(p)-\mathbf{X}(q).$$

Clearly $\mathbf{X}$ is an embedding if and only if

$$(\delta\mathbf{X})^{-1}(0)=D_{\mathbf{M}}=\{(p,p)\colon p\in\mathbf{M}\}\subset\mathbf{M}\times\mathbf{M}.$$

Since $\mathbf{X}$ is an immersion, it is locally injective, and there is an open neighborhood $U\subset\mathbf{M}\times\mathbf{M}$ of the diagonal $D_{\mathbf{M}}$ such that $\delta\mathbf{X}$ does not assume the value $0\in\mathbb{R}^n$ in $\overline{U}\setminus D_{\mathbf{M}}$.

To prove the theorem, it suffices to find arbitrarily close to $\mathbf{X}$ another conformal minimal immersion $\mathbf{Y}\colon\mathbf{M}\to\mathbb{R}^n$ whose difference map $\delta\mathbf{Y}$, restricted to $\mathbf{M}\times\mathbf{M}\setminus U$, is transverse to the origin $0\in\mathbb{R}^n$.

Since $\dim_{\mathbb{R}}\mathbf{M}\times\mathbf{M}=4<n$, this will imply that $\delta\mathbf{Y}$ does not assume the value $0\in\mathbb{R}^n$ in $\mathbf{M}\times\mathbf{M}\setminus U$, so $\mathbf{Y}(p)\neq\mathbf{Y}(q)$ if $(p,q)\in\mathbf{M}\times\mathbf{M}\setminus U$. It also does not assume the value $0\in\mathbb{R}^n$ on $U\setminus D_{\mathbf{M}}$ if $\mathbf{Y}$ is close enough to $\mathbf{X}$. Hence $\mathbf{Y}$ is an embedding.

Such a map $\mathbf{Y}$ is obtained by a transversality argument.

General Position Theorem - Proof

We begin by finding a holomorphic spray of maps $\mathbf{F}\colon B \times \mathbf{M} \to \mathbb{R}^n$, where $B \subset \mathbb{R}^N$ is a ball centered at $0 \in \mathbb{R}^N$, N large, such that

- $\mathbf{F}(0, \cdot) = \mathbf{X}$.
- $\mathbf{F}(x, \cdot)\colon \mathbf{M} \to \mathbb{R}^n$ is a conformal minimal immersion for all $x \in B$.
- The difference map

$$\delta\mathbf{F}\colon B \times \mathbf{M} \times \mathbf{M} \to \mathbb{R}^n, \quad \delta\mathbf{F}(x, p, q) = \mathbf{F}(x, p) - \mathbf{F}(x, q)$$

  is a submersive family on $\mathbf{M} \times \mathbf{M} \setminus U$, meaning that

$$d_x|_{x=0}\delta\mathbf{F}(x, p, q)\colon \mathbb{R}^N \to \mathbb{R}^n$$

  is surjective for every $(p, q) \in \mathbf{M} \times \mathbf{M} \setminus U$.

The standard transversality argument due to **Abraham** shows that for a generic choice of $x \in B' \Subset B$, the difference map $\delta\mathbf{F}(x, \cdot, \cdot)$ is transverse to $\{0\} \subset \mathbb{R}^n$ on $\mathbf{M} \times \mathbf{M} \setminus U$, and so it omits the value $0 \in \mathbb{R}^n$ by dimensions reasons.
Choose $\mathbf{Y} := \mathbf{F}(x, \cdot)$ for a suitable $x \in \mathbb{R}^N$ close to 0.

Proper minimal surfaces with given conformal structure

Theorem

Let $\mathbf{M}$ *be an open Riemann surface and* $n \geq 3$*.*

There is a conformal minimal immersion $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \colon \mathbf{M} \to \mathbb{R}^n$ *such that* $(\mathbf{X}_1, \mathbf{X}_2) \colon \mathbf{M} \to \mathbb{R}^2$ *is a proper map.*

Furthermore, if $n \geq 5$ *then* $\mathbf{X}$ *may be chosen to be an embedding.*

Corollary

Every open Riemann surface is the underlying conformal structure of a minimal surface in $\mathbb{R}^3$ *properly projecting into* $\mathbb{R}^2$*.*

1985 **Schoen-Yau's Conjecture** Every minimal surface in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$ is parabolic.

1980s **Sullivan's Conjecture** Every proper minimal surface in $\mathbb{R}^3$ of finite topology is parabolic. (Disproved by **Morales** in 2003, his counterexample was a conformal disc.)

Proper minimal surfaces with given conformal structure

Corollary

Every open Riemann surface properly embeds in $\mathbb{R}^5$ *as a conformal minimal surface.*

False in $\mathbb{R}^3$. There are even topological restrictions.

Open in $\mathbb{R}^4$. There are no topological restrictions.

Proper embeddings of manifolds

General problem: When is an abstract manifold of a certain kind embeddable as a submanifold of a Euclidean space?

- **Withney**: every smooth n-manifold embeds smoothly in $\mathbb{R}^{2n+1}$.
- **Nash, Gromov**: isometric immersions and embeddings of Riemannian manifolds in $\mathbb{R}^N$
- **Greene and Wu**: every Riemannian manifold of dimension n admits a harmonic embedding in $\mathbb{R}^{2n+1}$.
- **Remmert, Bishop, Narasimhan**: every n-dimensional Stein manifold embeds properly holomorphically in $\mathbb{C}^{2n+1}$. In particular, every open Riemann surface embeds properly in $\mathbb{C}^3$.
- **Eliashberg and Gromov, Schurmann**: Every Stein manifold of dimension $n > 1$ embeds in $\mathbb{C}^{[3n/2]+1}$ and this is sharp.

Open Problem: Does every open Riemann surface properly holomorphically embeds in $\mathbb{C}^2$? And in $\mathbb{R}^4$ as a conformal minimal surface? And by harmonic functions?

Proper minimal surfaces - Proof

We choose an exhaustion $\mathbf{M}_1 \Subset \mathbf{M}_2 \Subset \cdots \Subset \cup_{j=1}^{\infty} \mathbf{M}_j = \mathbf{M}$, such that every $\mathbf{M}_j$ is a smooth Runge domain in $\mathbf{M}$, an initial conformal minimal immersion $\mathbf{M}_1 \to \mathbb{R}^n$, and apply the following lemma in a recursive way.

Lemma

Let $U \Subset V$ be smoothly bounded compact Runge domains in $\mathbf{M}$. Let $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \colon U \to \mathbb{R}^n$ $(n \geq 3)$ be a conformal minimal immersion such that

$$\max\{|\mathbf{X}_1(p)|, |\mathbf{X}_2(p)|\} > \eta \quad \textit{for some } \eta > 0 \textit{ and all } p \in bU.$$

Then there is a conformal minimal immersion $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n) \colon V \to \mathbb{R}^n$ such that:

(I) *$\mathbf{Y}$ is as close as desired to $\mathbf{X}$ in the $\mathscr{C}^1(U)$ topology.*

(II) $\max\{|\mathbf{Y}_1(p)|, |\mathbf{Y}_2(p)|\} > \eta$ *for all $p \in V \setminus \mathring{U}$.*

(III) $\max\{|\mathbf{Y}_1(p)|, |\mathbf{Y}_2(p)|\} > \eta + 1$ *for all $p \in bV$.*

The limit map $\mathbf{M} \to \mathbb{R}^n$ solves the theorem. If $n \geq 5$ we use the general position theorem to get embeddings.

Proper minimal surfaces - Proof of the lemma

We will prove the lemma only for the noncritical case (i.e., U is a deformation retract of V); the critical case reduces to the noncritical one by a simple application of the Mergelyan theorem.

Note that $V \setminus \mathring{U} = \cup_{i=1}^{\mathfrak{i}} \mathcal{A}_i$ where the $\mathcal{A}_i$'s are pairwise disjoint compact annuli,

$$b\mathcal{A}_i = \alpha_i \cup \beta_i \quad \text{where } \alpha_i \subset bU,\ \beta_i \subset bV.$$

There exist $\mathfrak{j} \in \mathbb{N}$, subsets I_1 and I_2 of $I := \{1, \ldots, \mathfrak{i}\} \times \mathbb{Z}_{\mathfrak{j}}$ and compact connected subarcs $\{\alpha_{i,j} \colon (i,j) \in I\}$ satisfying the following conditions:

(a1) $\cup_{j=1}^{\mathfrak{j}} \alpha_{i,j} = \alpha_i$ for $i = 1, \ldots, \mathfrak{i}$.

(a2) $\alpha_{i,j}$ and $\alpha_{i,j+1}$ only meet in a common endpoint $p_{i,j}$.

(a3) $I_1 \cup I_2 = I$ and $I_1 \cap I_2 = \emptyset$.

(a4) $|\mathbf{X}_k(p)| > \eta$ for all $p \in \alpha_{i,j}$ and all $(i,j) \in I_k$, $k = 1, 2$.

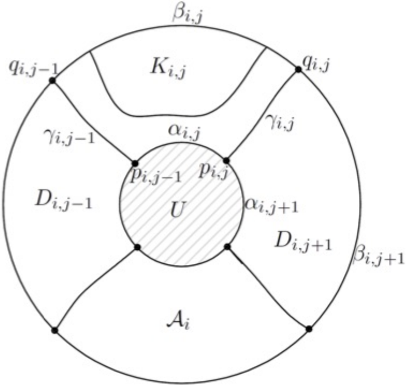
From (a4) one also has that

(a5) if $(i,j) \in I_h$ and $(i, j+1) \in I_l$, $h \neq l$, then $|\mathbf{X}_k(p_{i,j})| > \eta$ for $k \in \{1, 2\}$.

Proper minimal surfaces - Proof of the lemma

For every $(i,j) \in I$ we choose a smooth embedded arc $\gamma_{i,j} \subset \mathcal{A}_i$ with the following properties:

- $\gamma_{i,j}$ is attached to U at the endpoint $p_{i,j}$.
- $\gamma_{i,j}$ intersects the arc α_i transversely at $p_{i,j}$.
- $\gamma_{i,j} \cap \alpha_i = \{p_{i,j}\}$.
- The other endpoint $q_{i,j}$ of the arc $\gamma_{i,j}$ lies in $\beta_{i,j}$, $\gamma_{i,j}$ intersects β_i transversely at that point, and $\gamma_{i,j} \cap \beta_i = \{q_{i,j}\}$.
- The arcs $\gamma_{i,j}$, $(i,j) \in I$, are pairwise disjoint.



Proper minimal surfaces - Proof of the lemma

Extend $\mathbf{X}$ to a generalized conformal minimal immersion $(\mathbf{X}, f\vartheta)$ on the admissible set

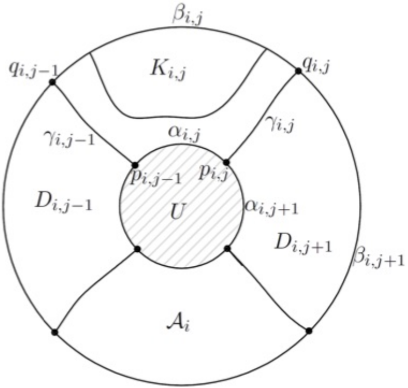
$$\mathbf{S} := U \cup \left(\cup_{(i,j)\in I}\, \gamma_{i,j} \cup \right) \Subset M$$

in such a way that:

(b1) $|\mathbf{X}_k(p)| > \eta$
for all $p \in \gamma_{i,j-1} \cup \gamma_{i,j}$,
$(i,j) \in I_k$, $k = 1, 2$.

(b2) $|\mathbf{X}_k(p)| > \eta + 1$
for all $p \in \{q_{i,j-1}, q_{i,j}\}$,
$(i,j) \in I_k$, $k = 1, 2$.

This is possible thanks to property (a4) above.



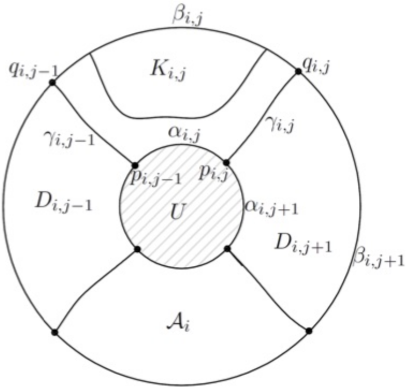
Proper minimal surfaces - Proof of the lemma

The Mergelyan theorem for CMIs gives a conformal minimal immersion $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n)\colon V \to \mathbb{R}^n$ satisfying the lemma on $\mathbf{S}$:

(c1) $\mathbf{F}$ is as close as desired to $\mathbf{X}$ in the $\mathscr{C}^1(U)$ topology.

(c2) $|\mathbf{F}_k(p)| > \eta$
for all $p \in \gamma_{i,j-1} \cup \alpha_i \cup \gamma_{i,j}$,
$(i,j) \in I_k$, $k = 1, 2$.

(c3) $|\mathbf{F}_k(p)| > \eta + 1$
for all $p \in \{q_{i,j-1}, q_{i,j}\}$,
$(i,j) \in I_k$, $k = 1, 2$.



(a4) and (b1) allow (c2).
(b2) allows (c3).
(c2) and (c3) give

(c4) if $(i,j) \in I_h$ and $(i,j+1) \in I_l$,
$h \neq l$, then for $k \in \{1, 2\}$ we have
$|\mathbf{F}_k(p)| > \eta$ for all $p \in \gamma_{i,j}$, and $|\mathbf{F}_k(q_{i,j})| > \eta + 1$.

Proper minimal surfaces - Proof of the lemma

Let $\beta_{i,j}$ be the Jordan arc in β_i which connects the points $q_{i,j-1}$ and $q_{i,j}$ and does not intersect the set $\{q_{i,h} \colon h \in \mathbb{Z}_j \setminus \{j-1, j\}\}$.

For every $(i,j) \in I$ we denote by $D_{i,j}$ the closed disc in $\mathcal{A}_i$ bounded bythe arcs $\gamma_{i,j-1}$, $\alpha_{i,j}$, $\gamma_{i,j}$, and $\beta_{i,j}$. Then $\mathcal{A}_i = \cup_{j=1}^{\mathrm{j}} D_{i,j}$ for every i.

By the continuity of $\mathbf{F}$, (c2), and (c3), there is for every $k \in \{1,2\}$ and every $(i,j) \in I_k$ a closed disc $K_{i,j} \subset D_{i,j} \setminus (\gamma_{i,j-1} \cup \alpha_{i,j} \cup \gamma_{i,j})$ such that:

(d1) $K_{i,j} \cap \beta_{i,j}$ is a compact connected Jordan arc.

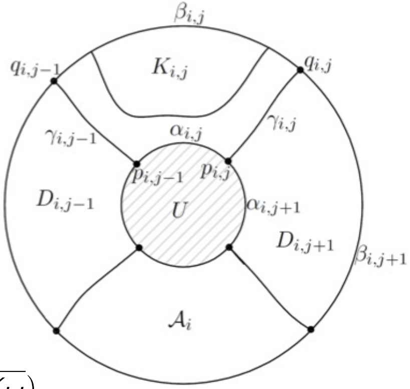
(d2) $|\mathbf{F}_k(p)| > \eta$ for all $p \in \overline{D_{i,j} \setminus K_{i,j}}$.

(d3) $|\mathbf{F}_k(p)| > \eta + 1$ for all $p \in \overline{\beta_{i,j} \setminus K_{i,j}}$.

Clearly we have

$V \setminus \mathring{U} = \cup_{(i,j)\in I}(K_{i,j} \cup \overline{D_{i,j} \setminus K_{i,j}})$,

$bV = \cup_{(i,j)\in I}((\beta_{i,j} \cap K_{i,j}) \cup \overline{\beta_{i,j} \setminus K_{i,j}})$.



Proper minimal surfaces - Proof of the lemma

Assume that $I_1 \neq \emptyset$; otherwise $I_2 = I \neq \emptyset$ and reason symmetrically.

We now deform $\mathbf{F}$ into a conformal minimal immersion $\mathbf{G}\colon V \to \mathbb{R}^n$ satisfying the lemma on the set $U \cup \left(\cup_{(i,j)\in I_1} D_{i,j} \right)$. So we perturb $\mathbf{F}$ largely in $\cup_{(i,j)\in I_1} K_{i,j}$ with a suitable control elsewhere.

Consider the admissible compact set

$$\mathbf{S}_1 := \Big(U \cup \left(\cup_{(i,j)\in I_2} D_{i,j} \right) \Big) \cup \Big(\cup_{(i,j)\in I_1} K_{i,j} \Big) \subset V.$$

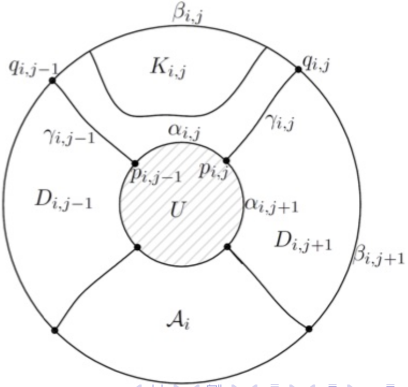
Note that the compact sets

$U \cup \left(\cup_{(i,j)\in I_2} D_{i,j} \right)$

and

$\cup_{(i,j)\in I_1} K_{i,j}$

are disjoint.



Proper minimal surfaces - Proof of the lemma

Choose $y_2 > 0$ such that

$$|y_2 + \mathbf{F}_2(p)| > \eta + 1 \quad \text{for all } p \in \cup_{(i,j)\in I_1} K_{i,j},$$

and define $\widetilde{\mathbf{G}} = (\widetilde{\mathbf{G}}_1, \widetilde{\mathbf{G}}_2, \ldots, \widetilde{\mathbf{G}}_n)\colon \mathbf{S}_1 \to \mathbb{R}^n$ by:

(e1) $\widetilde{\mathbf{G}}(p) = \mathbf{F}(p)$ for all $p \in U \cup \big(\cup_{(i,j)\in I_2} D_{i,j}\big)$.

(e2) $\widetilde{\mathbf{G}}(p) = (0, y_2, \ldots, 0) + \mathbf{F}(p)$ for all $p \in \cup_{(i,j)\in I_1} K_{i,j}$.

By using a special version of the Mergelyan theorem for CMIs, which enables one to preserve one component function whenever it extends harmonically, we obtain a conformal minimal immersion $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n)\colon V \to \mathbb{R}^n$ such that:

$$\mathbf{G}_1 = \mathbf{F}_1.$$

(f1) $\mathbf{G}$ is as close to $\widetilde{\mathbf{G}}$ as desired in the $\mathscr{C}^1(\mathbf{S}_1)$ topology.

(f2) $|\mathbf{G}_k(p)| > \eta$ for all $p \in \overline{D_{i,j} \setminus K_{i,j}}$, $(i,j) \in I_k$, $k = 1, 2$.

(f3) $|\mathbf{G}_k(p)| > \eta + 1$ for all $p \in \overline{\beta_{i,j} \setminus K_{i,j}}$, $(i,j) \in I_k$, $k = 1, 2$.

(f4) $|\mathbf{G}_2(p)| > \eta + 1$ for all $p \in K_{i,j}$ and all $(i,j) \in I_1 = I \setminus I_2$.

Proper minimal surfaces - Proof of the lemma

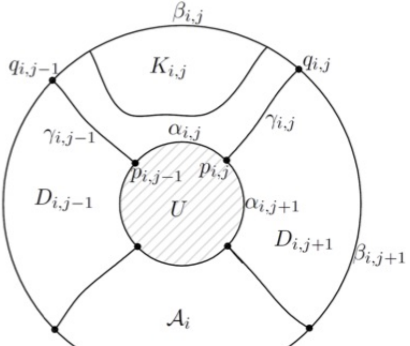
If $I_2 = \emptyset$ then

$$V = U \cup \big(\cup_{(i,j)\in I_1} K_{i,j} \big) \cup \big(\cup_{(i,j)\in I_1} \overline{D_{i,j} \setminus K_{i,j}} \big),$$

$$bV = \big(\cup_{(i,j)\in I_1} \beta_{i,j} \cap K_{i,j} \big) \cup \big(\cup_{(i,j)\in I_1} \overline{\beta_{i,j} \setminus K_{i,j}} \big),$$

and the Lemma already holds with $\mathbf{Y} = \mathbf{G}$.

Assume that $I_2 \neq \emptyset$. A deformation procedure which is symmetric to the one in the previous step (now the deformation is large in $\cup_{(i,j)\in I_2} K_{i,j}$ and controlled elsewhere) gives a conformal minimal immersion $\mathbf{Y}\colon V \to \mathbb{R}^n$ satisfying the conclusion of the lemma.



Minimal Surfaces and Complex Analysis

Lecture 2: Mergelyan approximation and applications

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