Minimal Surfaces and Complex Analysis

Lecture 3: Riemann-Hilbert problem and applications

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Lecture 3: Riemann-Hilbert problem and applications

In this lecture we will discuss the existence of approximate solutions to Riemann-Hilbert type boundary value problem for conformal minimal immersions $\mathbf{M} \to \mathbb{R}^n$ ($\mathbf{M}$ a bordered Riemann surface, $n \geq 3$), and use them to construct complete bounded minimal immersions $\mathbf{M} \to \mathbb{R}^n$, embeddings if $n \geq 5$.

Based on joint work with

- **Barbara Drinovec Drnovšek** and **Franc Forstnerič**, University of Ljubljana.
- **Francisco J. López**, University of Granada.

[A. Alarcón, F. Forstnerič: *Every bordered Riemann surface is a complete proper curve in a ball.* Math. Ann., 2013]
[A. Alarcón, F. Forstnerič: *The Calabi-Yau problem, null curves, and Bryant surfaces.* Math. Ann., in press]
[A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López: *Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves.* Proc. London Math. Soc., in press]

The classical Riemann-Hilbert problem

Let $f \colon \overline{\mathbb{D}} \to \mathbb{C}^n$ be a $\mathscr{C}^0$ map that is holomorphic in $\mathbb{D}$. Let

$$g \colon b\mathbb{D} \times \overline{\mathbb{D}} \to \mathbb{C}^n$$

be a $\mathscr{C}^0$ map such that $g(\xi, \cdot)$ is holomorphic in $\mathbb{D}$ and $g(\xi, 0) = f(\xi)$ for every $\xi \in b\mathbb{D}$.

Let $K \subset \mathbb{D}$ be a compact set and let $\epsilon > 0$.

Problem: Find a $\mathscr{C}^0$ map $\phi \colon \overline{\mathbb{D}} \to \mathbb{C}$, holomorphic in $\mathbb{D}$, such that:

- ϕ is ϵ-close to f over K.
- $\phi(\xi)$ is ϵ-close to the curve $g(\xi, b\overline{\mathbb{D}})$ for every $\xi \in b\mathbb{D}$.

The model case: $n = 2$, $f(\xi) = (\xi, 0)$ and $g(\xi, \zeta) = (\xi, \zeta)$. A solution is $\phi(\xi) = (\xi, \xi^N)$ for large $N \in \mathbb{N}$.

2007 **Drinovec Drnovšek and Forstnerič** Solutions exist even when the source manifold $\mathbb{D}$ is replaced by any bordered Riemann surface and the target manifold $\mathbb{C}^n$ by an arbitrary complex manifold.

The Riemann-Hilbert method is useful in a variety of problems; in particular for constructing proper curves.

Riemann-Hilbert method for conformal minimal immersions

Theorem

Let $\mathbf{M}$ *be a compact bordered Riemann surface and let* $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n$ $(n \geq 3)$ *be a conformal minimal immersion (the* **central surface***).*
Let I *be a compact subarc of* $b\mathbf{M}$ *which is not a connected component of* $b\mathbf{M}$*. Choose a small annular neighborhood* $A \subset \mathbf{M}$ *of the component* C *of* $b\mathbf{M}$ *containing* I *and a smooth retraction* $\rho\colon A \to C$*.*
Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *be a couple of unitary orthogonal vectors (the* **direction vectors***), let* $\mu\colon C \to \mathbb{R}_+$ *be a continuous function supported on* I *(the* **size function***), and consider the continuous map*

$$\varkappa\colon b\mathbf{M} \times \overline{\mathbb{D}} \to \mathbb{R}^n$$

$$\varkappa(x,\xi) = \begin{cases} \mathbf{X}(p); & p \in b\mathbf{M} \setminus I \\ \mathbf{X}(p) + \mu(p)\big(\Re\xi\mathbf{u} + \Im\xi\mathbf{v}\big), & p \in I. \end{cases}$$

Riemann-Hilbert method for conformal minimal immersions

Theorem (Continued)

Then for any number $\epsilon > 0$ there exist an arbitrarily small open neighborhood Ω of I in A and a conformal minimal immersion $\mathbf{Y}\colon \mathbf{M} \to \mathbb{R}^n$ satisfying the following properties:

- $\mathbf{Y}$ *is* ϵ*-close to* $\mathbf{X}$ *in the* $\mathcal{C}^1$ *topology on* $\mathbf{M} \setminus \Omega$.
- $\mathrm{dist}(\mathbf{Y}(p), \varkappa(p, b\mathbb{D})) < \epsilon$ *for all* $p \in b\mathbf{M}$.
- $\mathrm{dist}(\mathbf{Y}(p), \varkappa(\rho(p), \overline{\mathbb{D}})) < \epsilon$ *for all* $p \in \Omega$.
- $\mathrm{Flux}_{\mathbf{Y}} = \mathrm{Flux}_{\mathbf{X}}$.

- We do not change the conformal structure on **M**.
- I can be replaced by a finite family of pairwise disjoint compact subarcs; it is allowed to use different direction vectors in each subarc.
- The *boundary discs* can be arbitrary planar discs (non-necessarily round) in parallel planes, and in case $n = 3$ they can be arbitrary **minimal** discs (non-necessarily planar).

The spinor representation of the null quadric in $\mathbb{C}^3$

Recall the null quadric

$$\mathfrak{A}^* = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \colon \sum_{j=1}^n z_j^2 = 0\} \setminus \{0\}$$

directing conformal minimal immersions $\mathbf{M} \to \mathbb{R}^n$.

If $n = 3$, the complex cone $\mathfrak{A} = \mathfrak{A}^* \cup \{0\}$ admits a **spinor representation**:

$$\pi \colon \mathbb{C}^2 \to \mathfrak{A}, \quad \pi(a, b) = (a^2 - b^2, \imath(a^2 + b^2), 2ab).$$

The map

$$\pi \colon \mathbb{C}^2 \setminus \{(0,0)\} \to \mathfrak{A}^*$$

is a nonramified two-sheeted covering.

The Riemann-Hilbert method - Proof for $n = 3$

We first consider the case $\mathbf{M} = \overline{\mathbb{D}}$.

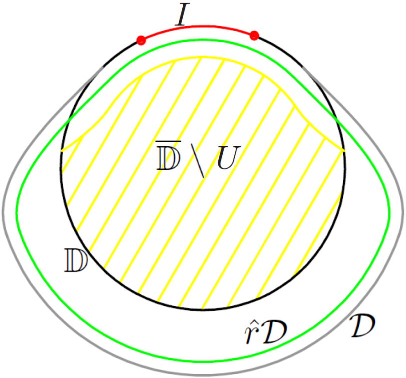
$$\mathfrak{A}^* = \{(a^2 - b^2, \imath(a^2 + b^2), 2ab) \in \mathbb{C}^3 \colon (a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}\}$$

$$\begin{aligned}
\mathbf{X}' &= \big(a^2 - b^2, \imath(a^2 + b^2), 2ab\big) \colon \overline{\mathbb{D}} \to \mathfrak{A}^* \subset \mathbb{C}^3 \\
\mathbf{u} - \imath\mathbf{v} &= \big(p^2 - q^2, \imath(p^2 + q^2), 2pq\big) \in \mathfrak{A}^* \\
\eta &= \sqrt{\mu} \colon b\mathbb{D} = \mathbb{S}^1 \to \mathbb{R}_+ \\
\eta(\zeta) &\approx \tilde{\eta}(\zeta) = \sum_{j=1}^{N} A_j \zeta^{j-m} \quad (\text{rational approximation on } \mathbb{C} \setminus \{0\}) \\
a_k(\xi) &= a(\xi) + \sqrt{2k+1}\,\tilde{\eta}(\xi)\,\xi^k p \quad (k > m,\ u_k(0) = u(0)) \\
b_k(\xi) &= b(\xi) + \sqrt{2k+1}\,\tilde{\eta}(\xi)\,\xi^k q \quad (v_k(0) = v(0)) \\
\Phi_k &= \big(a_k^2 - b_k^2, \imath(a_k^2 + b_k^2), 2a_k b_k\big) \colon \overline{\mathbb{D}} \to \mathfrak{A}^* \\
Y_k(\zeta) &= \mathbf{X}(0) + \mathrm{Re}\Big(\int_0^{\zeta} \Phi_k(\xi)\, d\xi\Big), \quad \zeta \in \overline{\mathbb{D}}.
\end{aligned}$$

It follows that $Y_k(\zeta) \approx \mathbf{X}(\zeta) + \mu(\zeta)\big(\mathrm{Re}(\zeta^{2k+1})\mathbf{u} + \mathrm{Im}(\zeta^{2k+1})\mathbf{v}\big)$. Take $\mathbf{Y} = Y_k$ for large enough $k \in \mathbb{N}$.

The Riemann-Hilbert method - Proof for $n = 3$

Furthermore, if I is a compact arc in $b\mathbb{D}$, the size function μ vanishes everywhere on $b\mathbb{D} \setminus I$, and U is an open neighborhood of I in $\overline{\mathbb{D}}$, then one can choose $\mathbf{Y}$ to be ϵ-close to $\mathbf{X}$ in the $\mathscr{C}^1$ topology on $\overline{\mathbb{D}} \setminus U$.



The Riemann-Hilbert method - Proof for $n = 3$

Assume now that $\mathbf{M}$ is any compact bordered Riemann surface.

- Solve the problem in a small disc $D \subset \Omega \subset \overline{\mathbf{M}} \setminus A$ containing I. (We have just proved that we may.) Call $\mathbf{Y}_0 \colon \overline{D} \to \mathbb{R}^3$ the solution.
- Let ϑ be a nowhere vanishing holomorphic 1-form on $\mathbf{M}$ and write

$$\partial \mathbf{Y}_0 = \mathbf{g}_0 \vartheta, \quad \mathbf{g}_0 \colon \overline{D} \to \mathfrak{A}^*,$$

$$\partial \mathbf{X} = \mathbf{f}_0 \vartheta, \quad \mathbf{f}_0 \colon \overline{\mathbf{M}} \to \mathfrak{A}^*.$$

  Observe that $\int_\gamma \mathbf{f}_0 \vartheta = \imath \mathrm{Flux}_{\mathbf{X}}(\gamma)$ for every $\gamma \in H_1(\mathbf{M}; \mathbb{Z})$.
- Embed $\mathbf{f}_0$ as the core map of a dominating and period-dominating holomorphic spray of holomorphic maps

$$\left\{\mathbf{f}_t \colon \overline{\mathbf{M}} \to \mathfrak{A}^*\right\}_{t \in B}.$$

- Embed $\mathbf{g}_0$ as the core map of a dominating spray of holomorphic maps

$$\left\{\mathbf{g}_t \colon \overline{D} \to \mathfrak{A}^*\right\}_{t \in B}, \quad 0 \in B \subset \mathbb{C}^N \quad \text{(large enough } N\text{)},$$

  such that $\mathbf{g}_t \vartheta$ provide by integration approximate solutions to the Riemann-Hilbert problem in $\overline{D}$.

The Riemann-Hilbert method - Proof for $n = 3$

- Glue the two (dominating) sprays $\{\mathbf{f}_t\}_{t\in B}$ and $\{\mathbf{g}_t\}_{t\in B}$ into a single spray of holomorphic maps
$$\{\mathbf{h}_t\colon \overline{\mathbf{M}} \to \mathfrak{A}^*\}_{t\in B}$$
such that $\mathbf{h}_t$ is close to $\mathbf{g}_t$ on $\overline{D}$ and to $\mathbf{f}_t$ on $\overline{\mathbf{M}\setminus D}$.
- Since $\mathbf{f}_0\theta$ has no real periods and the spray $\mathbf{f}_t$ is period dominating, so is $\mathbf{h}_t$ provided the approximation is close enough. Hence the period map
$$B \ni t \mapsto \mathcal{P}(\mathbf{h}_t) = \Big(\int_\gamma \mathbf{h}_t\vartheta\Big)_{\gamma\in H_1(\mathbf{M};\mathbb{Z})}$$
has maximal rank at $t = 0$. The Implicit Function Theorem gives $t_0 \in B$ close to 0 such that $\int_\gamma \mathbf{h}_{t_0}\vartheta = \int_\gamma \mathbf{f}_0\vartheta = \imath\mathrm{Flux}_{\mathbf{X}}(\gamma)$ for every $\gamma \in H_1(\mathbf{M};\mathbb{Z})$
- Fix a point $p_0 \in \mathbf{M}\setminus D$. The map $\mathbf{Y}\colon \overline{\mathbf{M}} \to \mathbb{R}^3$,
$$\mathbf{Y}(p) = \mathbf{X}(p_0) + \Re\int_{p_0}^{p} \mathbf{h}_{t_0}\vartheta$$
proves the theorem.

The Riemann-Hilbert method - Proof

The proof in general dimension $n \geq 4$ consists on reducing the problem to dimension 3.

The proof for non-round boundary disc (arbitrary minimal discs if $n = 3$) uses conformal parameterizations.

Calabi's Conjecture

1963 **Calabi's Conjecture** Complete nonplanar minimal surfaces in $\mathbb{R}^3$ have no bounded coordinate function.

In particular, there is no complete bounded minimal surface in $\mathbb{R}^3$.

1980 **Jorge-Xavier** There exists a complete minimal surface contained in a slab of $\mathbb{R}^3$.

1996 **Nadirashvili** There exists a complete minimal surface contained in a ball of $\mathbb{R}^3$.

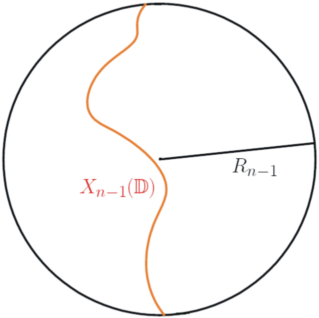
Nadirashvili's technique

$X_n : \mathbb{D} \to \mathbb{B}_{R_n}$ conformal minimal immersion

- $\|X_n - X_{n-1}\| \approx 0$ in $\overline{\mathbb{D}}_{1-1/n}$.
- $\mathrm{dist}_{ds_{X_n}}(0, \partial\mathbb{D}) \approx \sum_{k=1}^{n} \frac{1}{k}$.
- $R_n \approx \sqrt{\sum_{k=1}^{n} \frac{1}{k^2}}$.

$\Downarrow$

$\{X_n\} \to X : \mathbb{D} \to \mathbb{R}^3$
complete bounded
minimal immersion



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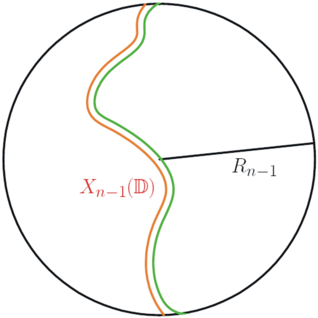

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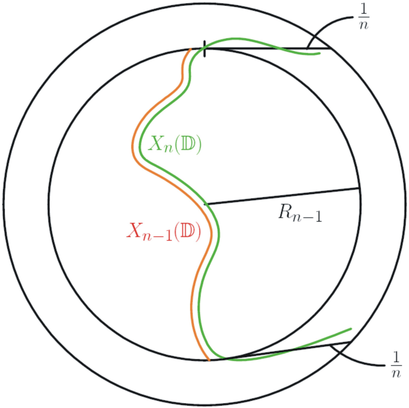

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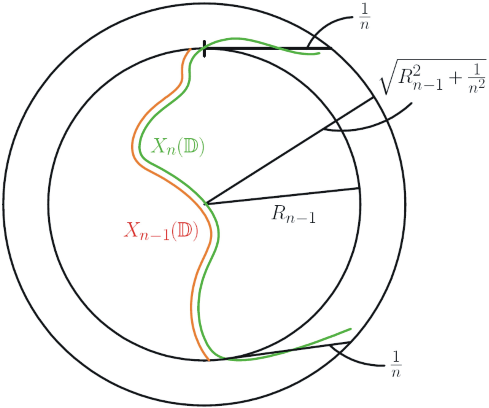
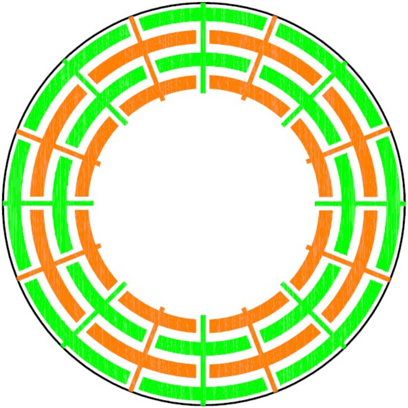


Nadirashvili's technique

Key tools: Runge's Theorem and the **López-Ros transformation** for conformal minimal immersions:

$$\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \rightsquigarrow (g, \phi_3) \mapsto (hg, \phi_3) \rightsquigarrow \mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3 = \mathbf{X}_3)$$



The Calabi-Yau problem

2000 **Yau** What is the geometry of complete bounded minimal surfaces in $\mathbb{R}^3$?

Jorge-Xavier's and Nadirashvili's method do not provide any control on the self-intersections of the examples.

2008 **Colding-Minicozzi** A complete finitely-connected **embedded** minimal surface in $\mathbb{R}^3$ must be proper in $\mathbb{R}^3$.

Meeks-Pérez-Ros Extension for surfaces of finite genus and countably many ends.

Nadirashvili's method does not provide any information on the conjugate surface of the examples.

Theorem

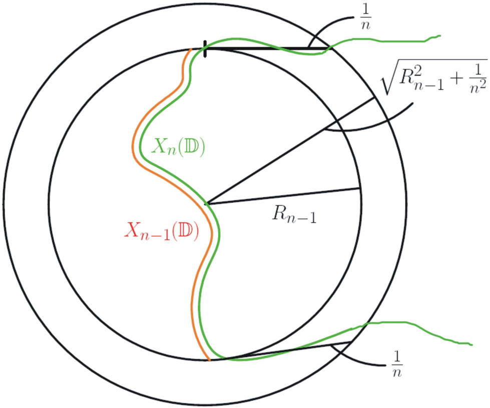
There exist complete bounded embedded **null holomorphic curves** *in* $\mathbb{C}^3$*.*

[A. Alarcón, F.J. López: *Null curves in* $\mathbb{C}^3$ *and Calabi-Yau conjectures.* Math. Ann. 2013]
[A. Alarcón, F. Forstnerič: *Null curves and directed immersions of open Riemann surfaces.* Invent. Math. 2014]

The conformal Calabi-Yau problem

Nadirashvili's method works for surfaces of finite topology

2012 **Ferrer-Martín-Meeks** There exist complete bounded minimal surfaces in $\mathbb{R}^3$ with **arbitrary topology**.



The conformal Calabi-Yau problem

Nadirashvili's method works for surfaces of finite topology, but it does **not** enable one to **control the conformal structure** of the examples.

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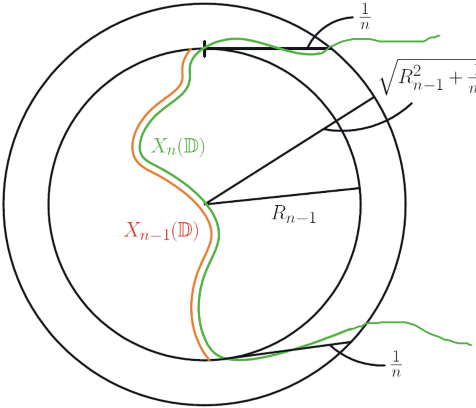

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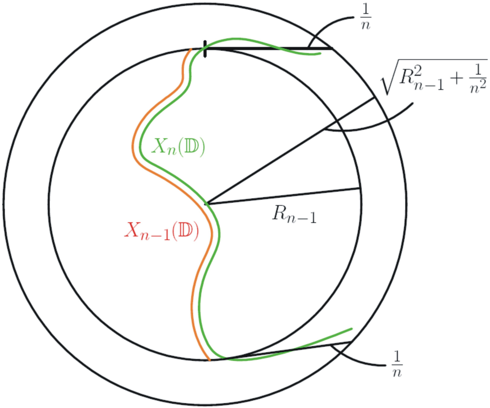


The conformal Calabi-Yau problem

Q. Which open Riemann surfaces are the conformal structure of a complete bounded minimal surface in $\mathbb{R}^3$?

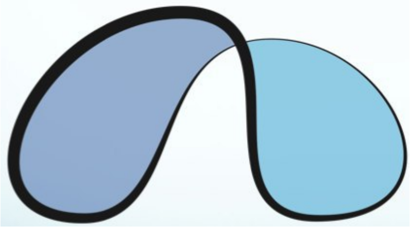
Theorem

Every bordered Riemann surface *carries a complete bounded null holomorphic immersion into* $\mathbb{C}^3$ *and hence a conformal complete minimal immersion into* $\mathbb{R}^3$ *with bounded image.*

[A. Alarcón, F. Forstnerič: *Null curves and directed immersions of open Riemann surfaces.* Math. Ann., in press]

Local theory: Plateau Problem

1873 **Plateau** Minimal surfaces can be physically obtained as soap films.



1931 **Douglas, Radó** Every continuous injective closed (i.e. Jordan) curve in $\mathbb{R}^n$ $(n \geq 3)$ spans a minimal surface.

The asymptotic Calabi-Yau problem

There is not much information about the (global) properties of solutions to Plateau Problems.
The solution surface for a rectifiable Jordan curve is NOT complete by the isoperimetric inequality for minimal surfaces.

Nadirashvili's method does not provide any information on the asymptotic behavior of the examples.

Q. **Are there complete minimal surfaces in $\mathbb{R}^3$ bounded by Jordan curves?**

Equivalently,

Are there Jordan curves in $\mathbb{R}^3$ spanning complete minimal surfaces?

Q. **Which domains in $\mathbb{R}^3$ are the natural containers of complete proper minimal surfaces?**

Higher dimension

Nadirashvili's method does not apply for surfaces in $\mathbb{R}^n$, $n \geq 4$.
(The López-Ros transformation is not available.)

Main Theorem

Theorem

Let $\mathbf{M} = \mathring{\mathbf{M}} \cup b\mathbf{M}$ *be a compact* **bordered Riemann surface**.

Every conformal minimal immersion $\mathbf{X}\colon \mathbf{M} \to \mathbb{R}^n$ $(n \geq 3)$ *of class* $\mathcal{C}^1(\mathbf{M})$ *can be uniformly approximated in the* $\mathcal{C}^0(\mathbf{M})$*-topology by continuous maps* $\widetilde{\mathbf{X}}\colon \mathbf{M} \to \mathbb{R}^n$ *such that:*

- $\widetilde{\mathbf{X}}|_{\mathring{\mathbf{M}}}\colon \mathring{\mathbf{M}} \to \mathbb{R}^n$ *is a conformal* **complete** *minimal immersion (with bounded image).*
- $\widetilde{\mathbf{X}}|_{b\mathbf{M}}\colon b\mathbf{M} \to \mathbb{R}^n$ *is an embedding. In particular,* $\widetilde{\mathbf{X}}(b\mathbf{M}) \subset \mathbb{R}^n$ *is a finite collection of pairwise disjoint* **Jordan curves**.

Furthermore, if $n \geq 5$ *then* $\widetilde{\mathbf{X}}$ *can be taken to be an embedding.*

This is an existence result.

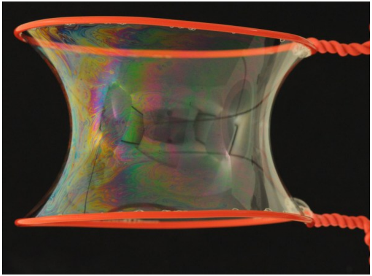
[A. Alarcón, B. Drinovec Drnovšek, F. Forstnerič, F.J. López: *Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves.* Preprint 2015]

Our result

Corollary

Every finite collection of pairwise disjoint Jordan curves in $\mathbb{R}^n$ admitting a connected solution to the Plateau Problem *also admits* **approximate solutions** *by* **complete** *minimal surfaces.*

Not every finite family of Jordan curves admits a connected solution to the Plateau problem.



Proof of the Theorem

Lemma (Key Lemma)

Let $\mathbf{M}$ *be a compact bordered Riemann surface. Let* $\mathbf{X} : \mathbf{M} \to \mathbb{R}^n$ $(n \geq 3)$ *be a conformal minimal immersion of class* $\mathcal{C}^1(\mathbf{M})$*, let* $\mathfrak{f} \colon b\mathbf{M} \to \mathbb{R}^n$ *be a smooth map, and let* $\delta > 0$ *be a number. Assume that*

$$\|\mathbf{X} - \mathfrak{f}\|_{0,b\mathbf{M}} < \delta.$$

Fix a point $p_0 \in \mathring{\mathbf{M}}$*.*

Then for each $\eta > 0$ *the immersion* $\mathbf{X}$ *can be approximated uniformly on compacts in* $\mathring{\mathbf{M}}$ *by conformal minimal immersions* $\widetilde{\mathbf{X}} : \mathbf{M} \to \mathbb{R}^n$ *of class* $\mathcal{C}^1(\mathbf{M})$ *satisfying the following properties:*

(a) $\operatorname{dist}_{\widetilde{\mathbf{X}}}(p_0, b\mathbf{M}) > \operatorname{dist}_{\mathbf{X}}(p_0, b\mathbf{M}) + \eta$.

(b) $\|\widetilde{\mathbf{X}} - \mathfrak{f}\|_{0,b\mathbf{M}} < \sqrt{\delta^2 + \eta^2}$.

Proof of the Theorem via the Key Lemma

Applying the lemma in a (finite) recursive way to the data

$$\mathbf{X}_j, \quad \mathfrak{f} = \mathbf{X}|_{b\mathbf{M}}, \quad \eta = \frac{\epsilon}{j},$$

and taking into account the Maximum Principle, we get (note that $\sum_j \frac{1}{j} = +\infty$ and $\sum_j \frac{1}{j^2} < +\infty$.)

Lemma

Let $\mathbf{M}$ *be a compact bordered Riemann surface and* $p_0 \in \mathbf{M}$*.*

Every conformal minimal immersion $\mathbf{X} : \mathbf{M} \to \mathbb{R}^n$ $(n \geq 3)$ *of class* $\mathcal{C}^1(\mathbf{M})$ *can be approximated in the* $\mathcal{C}^0(\mathbf{M})$*-topology by conformal minimal immersions* $\widetilde{\mathbf{X}} : \mathbf{M} \to \mathbb{R}^n$ *of class* $\mathcal{C}^1(\mathbf{M})$ *such that* $\mathrm{dist}_{\widetilde{\mathbf{X}}}(p_0, b\mathbf{M})$ *is* **as large as desired**.

Main Theorem follows from a recursive application of this lemma (and the General Position Theorem if $n \geq 5$).

Proof of the Key Lemma

Lemma (Key Lemma)

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Fix a point $p_0 \in \mathring{\mathbf{M}}$*.*

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(a) $\mathrm{dist}_{\widetilde{\mathbf{X}}}(p_0, b\mathbf{M}) > \mathrm{dist}_{\mathbf{X}}(p_0, b\mathbf{M}) + \eta.$

(b) $\|\widetilde{\mathbf{X}} - \mathfrak{f}\|_{0,b\mathbf{M}} < \sqrt{\delta^2 + \eta^2}.$

Proof of the Key Lemma

- By general position we may assume that

$$\mathbf{X}(p) - \mathfrak{f}(p) \neq 0 \quad \text{for all } p \in b\mathbf{M}.$$

- The key idea is to push the $\mathbf{X}$-image of each point $p \in b\mathbf{M}$ a distance approximately η in a direction approximately orthogonal to the vector $\mathbf{X}(p) - \mathfrak{f}(p) \in \mathbb{R}^n$. Conditions (a) and (b) will then follow from Pythagoras' Theorem.

- However, this procedure by itself will likely create shortcuts in the new induced metric. Hence we divide $b\mathbf{M}$ to finitely many very short arcs $I_1, \ldots, I_k$ so that both $\mathfrak{f}$ and $\mathbf{X}$ vary very little on each I_j when compared to the size of η (the desired displacement).

- At each of the endpoints $x_j = \mathbf{X}(p_j)$ of these arcs we attach to $\mathbf{X}(\mathbf{M}) \subset \mathbb{R}^n$ a smooth arc λ_j which remains near x_j, but is spinning fast and has long projection on each line spanned by one of the vectors $\mathbf{X}(p_i) - \mathfrak{f}(p_i)$.

Proof of the Key Lemma

- Using the Mergelyan theorem for conformal minimal immersions and the method of **exposing boundary points** by **Forstnerič and Wold**, we modify $\mathbf{X}$ so that it follows the arc λ_j and $\mathbf{X}(p_j) = q_j$ is the other endpoint of λ_j. Hence any curve in $\mathbf{M}$ terminating on $b\mathbf{M}$ near p_j is elongated a lot, at least more than η.
- To this new $\mathbf{X}$ we apply the Riemann-Hilbert method to find a conformal minimal immersion $\widetilde{\mathbf{X}}$ which at a interior point $x \in I_j$ adds a displacement for approximately η in a direction approximately orthogonal to the vector $\mathbf{X}(p_j) - \mathfrak{f}(p_j) \in \mathbb{R}^n$.
- The intrinsic boundary distance in $\widetilde{\mathbf{X}}(\mathbf{M})$ increases by approximately η, proving (a), whereas by Pythagoras

$$|\widetilde{\mathbf{X}}(p) - \mathfrak{f}(p)| \approx \sqrt{|\mathbf{X}(p) - \mathfrak{f}(p)|^2 + \eta^2} \leq \sqrt{\delta^2 + \eta^2} \quad \text{for all } p \in b\mathbf{M}.$$

  This bound also holds for all $p \in \mathbf{M}$ by the Maximum Principle, which proves (b).

Proper minimal surfaces in convex domains

The same tools are used to prove the following results on proper complete conformal minimal immersions.

Theorem

Let D be a convex domain in $\mathbb{R}^n$ for asome $n \geq 3$, and let $\mathbf{M}$ be a compact bordered Riemann surface.

(a) *Every conformal minimal immersion $\mathbf{X}\colon \mathbf{M} \to D$ of class $\mathscr{C}^1(\mathbf{M})$ can be approximated, uniformly on compacts in $\mathring{\mathbf{M}} = \mathbf{M} \setminus b\mathbf{M}$, by conformal complete proper minimal immersions $\widetilde{\mathbf{X}}\colon \mathbf{M} \to D$.*

(b) *If $n \geq 5$ then $\widetilde{\mathbf{X}}$ can be chosen an embedding.*

(c) *If D has smooth strongly convex boundary then $\widetilde{\mathbf{X}}$ can be chosen continuous on $\mathbf{M}$.*

In the proof we alternately apply the above Lemma (to enlarge the intrinsic boundary distance) and the Riemann-Hilbert method.

2012 **Ferrer-Martín-Meeks** Every convex domain in $\mathbb{R}^3$ carries complete proper minimal surfaces with arbitrary topology.

Mean-convex domains in $\mathbb{R}^3$

Let D be a domain in $\mathbb{R}^3$ with $\mathscr{C}^2$ boundary. Denote by $\kappa_1(x)$ and $\kappa_2(x)$ the principal curvatures of bD from the interior side at $x \in bD$.

Definition

A domain $D \subset \mathbb{R}^3$ with $\mathscr{C}^2$ boundary is said to be **mean convex** if $\kappa_1(x) + \kappa_2(x) \geq 0$ holds for every $x \in bD$. The domain D is **strongly mean convex** if $\kappa_1(x) + \kappa_2(x) > 0$ for every $x \in bD$.

Example: $\mathscr{C}^2$ convex domains are mean-convex but the converse is not true.

Example: A domain $D \subset \mathbb{R}^3$ bounded by an embedded minimal surface $\Sigma = bD \subset \mathbb{R}^3$ is mean-convex since in this case $\kappa_1(x) + \kappa_2(x) = 0$ for every $x \in \Sigma$.

Complete proper minimal surfaces in mean-convex domains

Theorem

Let D be a mean-convex domain in $\mathbb{R}^3$ with $\mathscr{C}^2$ boundary and let $\mathbf{M}$ be a compact bordered Riemann surface.

Every conformal minimal immersion $\mathbf{X}\colon \mathbf{M} \to D$ of class $\mathscr{C}^1(\mathbf{M})$ can be approximated, uniformly on compacts in $\mathring{\mathbf{M}} = \mathbf{M} \setminus b\mathbf{M}$, by conformal complete proper minimal immersions $\widetilde{\mathbf{X}}\colon \mathring{\mathbf{M}} \to D$.

If D is bounded and strongly mean-convex then $\widetilde{\mathbf{X}}$ can be chosen continuous on $\mathbf{M}$.

This is the first general existence result for complete proper minimal surface in domains in $\mathbb{R}^3$ which are not convex.

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Minimal Surfaces and Complex Analysis

Lecture 3: Riemann-Hilbert problem and applications

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