DEGENERATE COMPLEX MONGE-AMPÈRE EQUATIONS

Abstract. We introduce a new tool in plutipotential theory, the generalized Monge-Ampère capacities and use these to study degenerate complex Monge-Ampère equations whose right-hand side is smooth outside a divisor establishing uniform estimates which generalize both Yau's and Kołodziej's celebrated estimates.

We will also discuss as such generalized capacities turn out to be the key ingredient to show that the Kaehler-Ricci flow can be run from any arbitrary positive closed current, and that it is immediately smooth in a Zariski open subset of X.

All the results we will talk about are joint with C.H. Lu.

Let X be an n-dimensional compact Kähler manifold and fix ω an arbitrary Kähler form. We denote by $\mathrm{Ric}(\omega)$ the Ricci form of ω that is a closed positive $(1,1)$-form on X and it represents the first Chern class $c_1(X)$ of the underlying manifold. Conversely, given η a closed differential form representing $c_1(X)$, Calabi asked in [1] whether one can find a Kähler form ω such that

$$\mathrm{Ric}(\omega) = \eta.$$

He showed that if the answer is positive, then the solution is unique and he proposed a continuity method to prove the existence. This problem, known as the Calabi conjecture, remained open for two decades and it was finally solved by Yau in [16]. This result is now known as the Calabi-Yau theorem.

The Calabi conjecture reduces to solving a complex Monge-Ampère of type

$$(\omega + dd^c\varphi)^n = e^h\omega^n, \tag{CY}$$

where $h \in C^\infty(X,\mathbb{R})$. Note that h necessarily satisfies the normalizing condition

$$\int_X e^h\omega^n = \int_X \omega^n.$$

Theorem 0.1 (Yau78)**.** *The equation (CY) admits a unique (up to an additive constant) solution $\varphi \in C^\infty(X,\mathbb{R})$ such that $\omega_\varphi := \omega + dd^c\varphi$ is a Kähler form.*

Yau's proof relies on the *continuity method*, a classical tool to solve non linear PDE's. The goal is to establish various *a priori estimates*: in particular it suffices to prove C^0 and C^2-estimates. Indeed, thanks to Evans-Krylov theory we can deduce an estimate of type $C^{2,\alpha}$ and this suffices to apply Schauder's theorem and a bootstrap argument in order to conclude. The most difficult step are the C^0-estimates and Yau's approach uses Moser's iterative process. After the celebrated paper of Yau [16], Kołodziej [9] generalized the C^0 a priori estimates using pluripotential tools. His uniform estimate can indeed be applied to complex Monge-Ampère equations of the type

$$(\omega + dd^c\varphi)^n = f dV$$

where $0 \le f \in L^p(dV)$ for some $p > 1$. Under these weaker assumptions, he proves that the solution φ is globally bounded on X. Kołodziej's proof uses pluripotential methods. In order to give an idea of his approach, let me introduce the Monge-Ampère capacity of a Borel set $E \subset X$, defined as

$$\mathrm{Cap}(E) := \sup\left\{ \int_E (\omega + dd^c u)^n \ \mid\ u \in PSH(X,\omega) \quad -1 \leq u \leq 0 \right\}.$$

The Monge-Ampère capacity is zero for *small set* for the Monge-Ampère operator. Precisely, $\mathrm{Cap}(E) = 0$ if and only if E is pluripolar. The key point is then to prove that the function

$$H(t) := \mathrm{Cap}(\{\varphi < -t\})^{1/n}$$

satisfies

$$sH(t+s) \leq AH(t)^2 \quad \forall\, t > 0,\ \forall\, s \in (0,1). \tag{0.1}$$

Such an inequality allows indeed to deduce that there exists T_∞ such that the capacity of the sublevel set $(\varphi < -t)$ vanishes if $t \geq T_\infty$ and therefore that there exists $C > 0$ such that $\varphi \geq -C$.

Consider now a complex Monge-Ampère equation of the type

$$(\omega + dd^c\varphi)^n = f\omega^n, \tag{0.2}$$

where $f \in L^1(X)$ is such that $\int_X f\omega^n = \int_X \omega^n$. It is very natural for various geometric reasons to look at the case when f is merely smooth and positive on the complement of a divisor D, e.g. when studying Calabi's conjecture on quasiprojective manifolds (see e.g. [12, 13]). Note that such degenerate equations naturally appear when dealing with the problem of the existence of singular Kähler-Einstein metrics on varieties with mild singularities.
We recall that the existence and the uniqueness of a *weak* solution ($\varphi \in \mathcal{E}(X,\omega)$) of the equation (0.2) were proved in the last years by Guedj and Zeriahi [7] and Dinew [3], respectively. Thus the relevant question was about the regularity of the solution φ.

In [4] and [5] Chinh H. Lu and I study such a problem. In this wilder setting classical PDE's methods break down, and we found another approach using pluripotential theory. The first very general result that we were able to prove is the following:

Theorem 0.2 (Di Nezza - Lu 2014)**.** *Assume that $f \lesssim e^{-\phi}$, for some quasi-plurisubharmonic (qpsh for short) function ϕ. Let $\varphi \in \mathcal{E}(X,\omega)$ be the solution of (0.2) normalized such that $\sup_X \varphi = 0$. Then, for any $a > 0$ small enough (i.e. a is such that $a\phi \in PSH(X,\omega/2)$) there exists $A > 0$ depending only on $\int_X e^{2\varphi/a}\omega^n$ such that*

$$\varphi \geq a\phi - A. \tag{0.3}$$

Here, we want to stress that $\int_X e^{2\varphi/a}\omega^n$ is finite thanks to Skoda's theorem since any qpsh function belonging to the class $\mathcal{E}$ has zero Lelong number at each point.
The proof of the above theorem deeply relies on pluripotential methods: we follow Kołodziej's approach with various novelties. We should emphasise that in our case the solution is not bounded and therefore a natural idea is to bound the solution from below by a "model" quasi-plurisubharmonic function that can be also very singular. This is the reason why we introduce a new tool in pluripotential theory,

the *generalized Monge-Ampère capacities*, defined as

$$\mathrm{Cap}_\psi(E) := \sup\left\{ \int_E (\omega + dd^c u)^n \;\mid\; u \in PSH(X,\omega)\;\; \psi - 1 \le u \le \psi \right\}, \quad \forall E \subset X,$$

where ψ is a $\omega/2$-psh function.
The idea to prove the generalized C^0-estimate in (0.3) is then to show a generalized version of inequality 0.1 with $H(t) := \mathrm{Cap}_\psi(\{\varphi < \psi - t\})^{1/n}$ and $\psi = a\phi$. This implies that the generalized capacity of sublevel sets $(\varphi < \psi - t)$ vanishes when $t > 0$ is large enough. The lower bound in Theorem 0.2 is the key step that allows us to prove the following regularity result:

Theorem 0.3 (Di Nezza - Lu 2014)**.** *Assume that* $0 < f \in C^\infty(X \setminus D)$ *where* D *is a closed subset of* X*. Moreover, assume that* f *can be written of the form* $f = e^{\psi^+ - \psi^-}$*, where* $\psi^\pm$ *are qpsh funtions on* X *and* $\psi^- \in L^\infty_{loc}(X \setminus D)$*. Let* $\varphi \in \mathcal{E}(X,\omega)$ *be the solution of (0.2) normalized such that* $\sup_X \varphi = 0$*. Then* φ *is smooth outside* D*.*

The idea of the proof goes as follows:
Step 1. We use Demailly's regularization theorem to obtain quasi-decreasing sequences of smooth qpsh functions $\psi^\pm_\varepsilon$ converging to $\psi^\pm$. Then we let $\varphi_\varepsilon \in C^\infty(X)$ be the normalized ($\sup_X \varphi_\varepsilon = 0$) solution of the Monge-Ampère equation

$$(\omega + dd^c \varphi_\varepsilon)^n = c_\varepsilon e^{{\psi^+}_\varepsilon - {\psi^-}_\varepsilon}$$

where c_ε is a normalization constant. Observe that here we use Yau's theorem!
Step 2. The goal is to establish C^0 and C^2 a priori estimates for the sequence of smooth ω-psh functions (φ_ε).
Step 3. (C^0-estimates) We use Theorem 0.2 with $\phi = \psi^-$ to obtain *generalized* C^0-estimates.
Step 4. (C^2-estimates) Thanks to step 3 (the crucial step!) we are able to prove laplacian estimates of type

$$\Delta_\omega \varphi_\varepsilon \le C e^{-\psi^-}.$$

The generalized Monge-Ampère capacities are not just technical tools but, as showed above, they are the key ingredient when dealing with Monge-Ampère equations with degenerate right-hand side. Another application of such tools is to study the smoothing properties of the Kähler-Ricci flow running from a degenerate initial data. Let me briefly introduce the setting and the problem.
Let $\alpha_0 \in H^{1,1}(X,\mathbb{R})$ be a Kähler class. Fix $\omega_0 \in \alpha_0$ a Kähler form. We say that a family of Kähler metrics $\omega_t := \omega(t)$ solves the Kähler-Ricci flow (KRF for short) starting from ω_0 if

$$\text{(KRF)} \qquad \frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$$

and $\omega(0) = \omega_0$.
The Kähler-Ricci flow became one of the major tools in Kähler geometry through the work of many authors starting from Cao [2] who proved that the Kähler-Ricci flow on a compact Kähler manifold with $c_1(X) \le 0$ converges to the unique Kähler-Einstein metric endowed by the manifold.

The existence and uniqueness of the Kähler-Ricci flow starting from any Kähler form is due to Cao [2], Tsuji [15] and Tian-Zhang [14]:

Theorem 0.4. *Let $\omega_0 \in \alpha_0$ be a Kähler form. Then there exists a unique family of Kähler metrics $(\omega(t))_{t\in[0,T_{\max})}$ satisfying (KRF) and $\omega(0) = \omega_0$ where*

$$T_{\max} := \sup\{t > 0 \,|\, \alpha_0 - tc_1(X) > 0\}$$

is the maximal time of existence of the flow.

In relation to the "analytic analogue" of the Minimal Model Program, recently proposed by Song and Tian, one need to start the KRF from a "degenerate" initial data rather than a Kähler form.
Observe that $T_{\max}$ does not depend on the initial data but only on its cohomology class α_0, so at least it makes sense asking whether one can start the flow from any positive closed $(1,1)$-current $T_0 \in \alpha_0$.

In this direction Song and Tian [11] proved that if $T_0 \in \alpha_0$ is a positive $(1,1)$-current with continuous potential, then there exists a unique family of Kähler metrics $(\omega(t))_{t\in(0,T_{\max})}$ satisfying (KRF) and such that $\omega(t)$ converges to ω_0 uniformly as t goes to zero.

Recently, Guedj and Zeriahi [8] proved that if $T_0 \in \alpha$ is a positive current with zero Lelong numbers at any point, $\nu(T_0, x) = 0 \; \forall x \in X$, then there exists a family of Kähler metrics $(\omega(t))_{t\in(0,T_{\max})}$ satisfying (KRF) and such that $\omega(t)$ converges to T_0 in the weak sense of currents as t goes to zero.
Observe that the above result insures that starting from any positive current with zero Lelong numbers, the KRF immediately smooths out.

Then, a natural question is: *what does it happen when T_0 has positive Lelong numbers?*
The result that we were able to prove with Lu [6] is the following:

Theorem 0.5 (Di Nezza - Lu 2015)**.** *Let $T_0 \in \alpha_0$ be any positive $(1,1)$-current such that $c(T_0) > \frac{1}{2T_{\max}}$. Then there exists a unique family of positive $(1,1)$-currents, $(\omega(t))_{t\in(0,T_{\max})}$ such that $\omega(t)$ is smooth on the Zariski open subset $X \setminus D_{k(t)}$ and here it solves (KRF) in the classical sense.*
Moreover, $\omega(t)$ converges to T_0 in the weak sense of currents as t goes to zero.

Here $c(T_0)$ denotes the critical exponent of integrability of T_0.
For any $t \in (0, T_{\max})$, the subset $D_{k(t)}$ in the statement of the Theorem is described as the Lelong superlevel set of T_0:

$$D_{k(t)} := \{x \in X \mid \nu(T_0, x) > k(t)\}$$

where the constant $k(t)$ depends only on t and it is decreasing to 0 as t goes to 0. Note that for any $s > 0$, D_s is an analytic subset of X [10].

It is well known that (KRF) can be reduced to solve a scalar parabolic complex Monge-Ampère equation of type

$$\text{(CMAE)} \qquad (\theta_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t}\omega^n$$

where $\theta_t = \omega - t\mathrm{Ric}(\omega)$ is assumed to be Kähler for all $t \in [0, T_{\max})$. Precisely, φ_t solves (CMAE) if and only if $\omega_t := \theta_t + dd^c\varphi_t$ solves (KRF). Thus, at the level of potentials, our Theorem 0.5 states as follow:

Theorem 0.6 (Di Nezza - Lu 2015)**.** *Let φ_0 be any ω-psh function such that $c(\varphi_0) > 1/2T_{\max}$. Then there exists a solution φ_t of (CMAE) that is smooth on $X \setminus D_{k(t)}$. Moreover, φ_t converges to φ_0 in the L^1-topology as t goes to zero.*

As for Theorem 0.3, the proof relies on a priori estimates. More precisely, the strategy will be to pick $(\varphi_{0,j})$ a smooth sequence of strictly ω-psh functions decreasing to φ_0 as $j \to +\infty$ and to consider $(\varphi_{t,j})$ the unique smooth flow running from $\varphi_{0,j}$ (observe that here we use Theorem 0.4!). The goal is to establish uniform estimates that will allow us to pass to the limit but we should always take into account the singular behavior of the initial data φ_0. Once again, the most difficult step is to deal with the C^0-estimate. The theory of generalized capacities play a key role in establishing the following C^0-estimate for the complex parabolic Monge-Ampère equation (CMAE):

Theorem 0.7 (Di Nezza - Lu 2015)**.** *Fix $\varepsilon > 0$ and $T < T_{\max}$. Then there exists a uniform constant $C > 0$ such that for $t \in [\varepsilon, T]$ the following holds*

$$\varphi_t \geq \left(1 - \frac{t}{2T}\right)\psi - C.$$

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