CR Geometry, Mappings into Spheres, and Sums-Of-Squares
Lecture I

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September 28, 2015

Outline of Lectures

General Theory.

- **Lecture I:** Introduction, Motivation, Basic CR Geometry.
- **Lecture II:** Abstract CR manifolds. Embedding problems.
- **Lecture III:** Baouendi-Treves Approximation. Extension of CR functions on embedded CR manifolds.

Levi Nondegenerate CR Hypersurfaces and their Mappings.

- **Lecture IV:** Normal forms. Bergman and Szegö Kernels.
- **Lecture V:** Pseudohermitian Geometry, Nondegenerate CR geometry. Geometry and Analysis of CR Mappings.
- **Lecture VI:** Mappings into Flat Models, Sums-Of-Squares, and the Gap Conjecture.

Outline - Lecture I

Biholomorphic equivalence of domains in $\mathbb{C}^m$.

Classification of domains in the complex plane $\mathbb{C}$ rests the following corner stone in complex analysis:

Riemann Mapping Theorem

Let $\Omega \subset \mathbb{C}$ be a simply connected domain with $\Omega \neq \mathbb{C}$. Then, there exists a biholomorphism $f\colon \Omega \to \mathbb{D}$, where $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ denotes the unit disk.

This is no longer true in higher dimensions.

Theorem 0 (Poincaré)

There is no biholomorphic mapping $f\colon \mathbb{D}^2 = \mathbb{D} \times \mathbb{D} \to \mathbb{B}_2$, where $\mathbb{D}^2 := \{(z, w)\colon |z| < 1,\ |w| < 1\}$ is the unit bidisk in $\mathbb{C}^2$ and $\mathbb{B}_2 := \{(z, w)\colon |z|^2 + |w|^2 < 1\}$ the unit ball.

Poincaré's proof of Theorem 0.

Suppose there exists a biholomorphism $f\colon \mathbb{D}^2 \to \mathbb{B}_2$.

$$\implies \exists \text{ isomorphism } \; f^c\colon \mathrm{Aut}(\mathbb{B}_2) \to \mathrm{Aut}(\mathbb{D}^2), \quad f^c\phi = f^{-1}\circ\phi\circ f.$$

Poincaré computed $\mathrm{Aut}(\mathbb{B}_2)$, and $\mathrm{Aut}(\mathbb{D}^2)$:

$$\mathrm{Aut}(\mathbb{B}_2) \cong SU(2,1)/\sim, \quad \mathrm{Aut}(\mathbb{D}^2) = (SU(1,1)/\sim)^2.$$

In particular,

$$\dim_{\mathbb{R}} \mathrm{Aut}(\mathbb{B}_2) = 8, \quad \dim_{\mathbb{R}} \mathrm{Aut}(\mathbb{D}^2) = 6,$$

which means they cannot be isomorphic. □

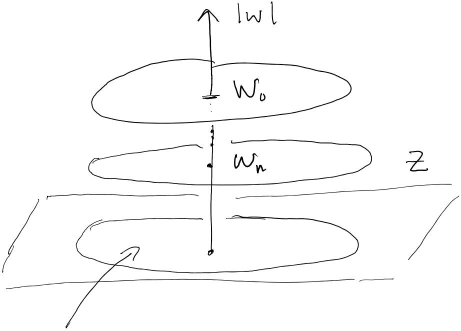
"CR" approach to Theorem 0.

Assume that there exists a biholomorphic (or just proper holomorphic) mapping $f\colon \mathbb{D}^2 \to \mathbb{B}_2$.

- Show that the holomorphic mapping induces a "partially holomorphic mapping" (CR) of the boundaries $f_0\colon \partial(\mathbb{D}^2) \to \partial\mathbb{B}_2$.
- Show that the boundaries have different "invariants" preserved by f_0; in this case, $\partial(\mathbb{D}^2)$ contains non-trivial complex curves, but $\partial\mathbb{B}_2$ does not.
- Conclude that no such mapping f can exists.

Bidisk $\mathbb{D}^2 \subseteq \mathbb{C}^2$

Sunday, August 30, 2015 6:32 AM



$\mathbb{D} = \{|w| < 1\}$

$\mathbb{D} = \{|z| < 1\}$

$$\partial(\mathbb{D}^2) = \{|z| < 1, |w| = 1\} \cup \{|z| = 1, |w| < 1\} \cup \{|z| = 1, |w| = 1\}$$

"CR" Proof of Theorem 0.

Suppose there exists a proper holomorphic mapping $f\colon \mathbb{D}^2 \to \mathbb{B}_2$;

$$f(z,w) = (f^1(z,w), f^2(z,w)).$$

Pick $w_0 \in \partial\mathbb{D}$, i.e., $|w_0| = 1$, and $w_n \in \mathbb{D}$ with $w_n \to w_0$. Set

$$A_n(z) = (A_n^1(z), A_n^2(z)) := (f^1(z,w_n), f^2(z,w_n)).$$

We note that $A_n^i(z)$ are holomorphic in $\mathbb{D}$, and $|A_n^i| \leq 1$. By Montel's Theorem, we may assume (by going to a subsequence) that there are holomorphic functions $A_0^i(z)$ in $\mathbb{D}$ such that $A_n^i \to A_0^i$.

Claim. $\|A_0(z)\|^2 := |A_0^1(z)|^2 + |A_0^2(z)|^2 = 1.$

Proof. By properness! $(z, w_n) \to \partial(\mathbb{D}^2)$. $\implies \|A_n(z)\|^2 \to 1.$ □

$\implies A_0\colon \mathbb{D} \to \partial\mathbb{B}_2$ is a holomorphic map (analytic disk).

"CR" Proof of Theorem 0; continued.

Lemma (No analytic disks in $\partial\mathbb{B}_2$)

If $A_0 \colon \mathbb{D} \to \partial\mathbb{B}_2$ is holomorphic, then $A_0(z)$ is constant.

Proof. Use $\mathrm{Aut}(\mathbb{B}_2)$! By replacing A_0 with UA_0, $U \in SU(2)$, we may assume that $A_0(0) = (1, 0)$.

- $\|A_0(z)\|^2 = 1 \implies |A_0^1(z)|$ has maximum at $z = 0$.
- Maximum Principle $\implies A_0^1(z)$ is constant, so $A_0^1(z) = 1$.
- $\|A_0(z)\|^2 = 1 \implies |A_0^2(z)|$ is identically 0.

□

We shall obtain a contradiction (proving Theorem 0) by showing

$$\frac{\partial f}{\partial z} = 0 \implies f \text{ not proper.}$$

End of proof of Theorem 0; $\partial f/\partial z = 0$.

Fix $z = z_0 \in \mathbb{D}$. Note that, for $j = 1, 2$,

$$\frac{\partial f^j}{\partial z}(z_0, w) = \frac{1}{2\pi i} \int_{|\zeta|=r<1} \frac{f^j(\zeta, w)d\zeta}{(\zeta - z_0)^2}$$

is bounded as a function of $w \in \mathbb{D}$. Thus, there are nontangential limits $h = (h^1, h^2)$, with $h^j \in L^\infty(\partial\mathbb{D})$, such that for a.e. $w_0 \in \partial\mathbb{D}$,

$$h(w_0) = \lim_{w \to w_0} \frac{\partial f}{\partial z}(z_0, w).$$

For $w_n \to w_0$ as before, we have $\partial f/\partial z(z_0, w_n) = A_n'(z_0) \to A_0'(z_0) = 0$ since $A_0(z)$ is constant. It follows that the nontangential limit at w_0 vanishes: $h(w_0) = 0$. Since this holds for all w_0 where the nontangential limits exist (a.e.), a standard uniqueness result implies

$$\frac{\partial f}{\partial z}(z_0, w) = 0.$$

□

Local biholomorphic equivalence of submanifolds.

Equivalence fails locally!

Proposition

Let $U \subset \mathbb{C}^2$ be an open neighborhood of $(z_0, w_0) \in \mathbb{C}^2$ with $|z_0| < 1$ and $|w_0| = 1$. If $f\colon U \to \mathbb{C}^2$ is a holomorphic mapping such that $f(\partial\mathbb{D}^2 \cap U) \subset \partial\mathbb{B}_2$, then $f = f(w)$.

Definition

Let $M_1, M_2 \subset \mathbb{C}^m$ be real submanifolds with $p_1 \in M_1$, $p_2 \in M_2$. If there exist an open neighborhood $U \subset \mathbb{C}^n$ of p_1 and a biholomorphic mapping $f\colon U \to f(U) \subset \mathbb{C}^n$ such that $f(p_1) = p_2$ and $f(M_1 \cap U) = M_2 \cap f(U)$, then (M_1, p_1) and (M_2, p_2) are said to be biholomorphically equivalent (BHE).

$$(M_1, p_1) \cong_{BHE} (M_2, p_2).$$

Remark: Different notions of equivalence! Analytic vs. smooth vs. formal.

Remarks on BHE.

- All real-analytic curves in $\mathbb{C}$ are locally BHE. $(\gamma, p) \cong_{BHE} (\mathbb{R}, 0)$.
- Real hypersurfaces in $\mathbb{C}^m$, $m \geq 2$, are in general not locally BHE. $(\partial\mathbb{D}^2, p_1) \ncong_{BHE} (\partial\mathbb{B}_2, p_2)$.
- For $m \geq 2$, a real hypersurface $M \subset \mathbb{C}^m$ is in general not BHE to *itself* at two different points. $(M, p_1) \ncong_{BHE} (M, p_2)$ if $p_1 \neq p_2$.
- But $\partial\mathbb{B}_m$ is. $(\partial\mathbb{B}_m, p_1) \cong_{BHE} (\partial\mathbb{B}_m, p_2)$, for all $p_1, p_2 \in \partial\mathbb{B}_m$. Such manifolds are called homogeneous.

Definition

A real-analytic hypersurface $M \subset \mathbb{C}^m$ is locally spherical at $p_1 \in M$ if $(M, p_1) \cong_{BHE} (\partial\mathbb{B}_m, p_2)$. (For smooth M use smooth CR equivalence.)

A Riemann Mapping Theorem in higher dimensions.

Theorem (S.-S. Chern – S. Ji, '96 [2])

Let $\Omega \subset \mathbb{C}^m$ be a bounded, simply connected domain. If $\partial\Omega$ is locally spherical, then there exists a biholomorphic mapping $f\colon \Omega \to \mathbb{B}_m$.

Remarks:

- If $\partial\Omega$ is not real-analytic, but smooth, then "locally spherical" at $p \in \partial\Omega$ can be defined as the existence of $p \in U \subset \mathbb{C}^n$ and a smooth mapping $f\colon U \cap \overline{\Omega} \to \mathbb{C}^n$ such that $f\colon U \cap \Omega \to f(U \cap \Omega)$ is biholomorphic and $f(\partial\Omega \cap U) \subset \partial\mathbb{B}_n$.
- In the real-analytic case, it in fact suffices that $\partial\Omega$ is locally spherical *at some point* $p \in \partial\Omega$. The local biholomorphism then extends as a global biholomorphism $f\colon \Omega \to \mathbb{B}_m$.
- X. Huang and S. Ji [4] have proved a Riemann Mapping Theorem for a more general class of domains, where again the assumption is that the boundaries are locally equivalent.

Recall [1]: Complex structure on a vector space.

A complex structure on $\mathbb{R}^{2m}$ is a linear map $J = J_p \colon T_p\mathbb{R}^{2m} \to T_p\mathbb{R}^{2m}$ such that $J^2 = -I$. J extends by linearity to $\mathbb{C}T_p\mathbb{R}^{2m} := \mathbb{C} \otimes T_p\mathbb{R}^{2m}$ and splits it into an $i = \sqrt{-1}$ and $-i$ eigenspace,

$$\mathbb{C}T_p\mathbb{R}^{2m} = T_p^{1,0}\mathbb{R}^{2m} \oplus T_p^{0,1}\mathbb{R}^{2m}$$

with $T_p^{0,1}\mathbb{R}^{2m} = \overline{T_p^{1,0}\mathbb{R}^{2m}}$. The standard complex structure in coordinates $(x_1, y_1, \ldots, x_m, y_m)$ is given by

$$J(\partial/\partial x_j) = \partial/\partial y_j, \quad J(\partial/\partial y_j) = -\partial/\partial x_j,$$

and $T_p^{1,0}\mathbb{R}^{2m}$ is spanned by $\partial/\partial z_1, \ldots, \partial/\partial z_m$,

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_j}\right).$$

The standard linear structure yields $\mathbb{C}^m$ with complex coordinates

$$z = (z_1, \ldots, z_m), \quad z_j = x_j + iy_j.$$

CR structure of a real hypersurface in a complex manifold.

Let $\Omega \subset \mathbb{C}^m$ be a domain with complex coordinate $z = (z_1, \dots, z_m)$. Let $M \subset \Omega$ be a real hypersurface; i.e., defined locally near every $p \in M$ by

$$M \cap V_p := \{z \in V_p \colon \rho(z, \bar{z}) = 0\},$$

where $p \in V_p \subset \Omega$, $\rho \in C^\kappa(V_p, \mathbb{R})$, $d\rho|_M \neq 0$. For us, κ is either ∞ ("smooth") or ω ("real-analytic").

Definition. The CR tangent space to M at $p \in M$ is given by

$$T_p^{0,1}M := \mathbb{C}T_pM \cap T_p^{0,1}\Omega; \quad T_p^{1,0}M := \overline{T_p^{0,1}M}.$$

$$L = \sum_{j=1}^m a_j \frac{\partial}{\partial z_j} \in T_p^{1,0}M \iff \sum_{j=1}^m \frac{\partial \rho}{\partial z_j}(p, \bar{p}) a_j = 0.$$

CR manifolds of hypersurface type.

- $T_p^{0,1}M$ is a complex hyperplane in the m-dimensional complex vector space $T_p^{0,1}\Omega$. Thus, $\dim_{\mathbb{C}} T_p^{0,1}M = m-1$ for all $p \in M$.
- Set $n = m-1$; $M \subset \Omega \subset \mathbb{C}^{n+1}$, $\dim_{\mathbb{R}} M = 2(n+1)-1 = 2n+1$.
- $T_p^{0,1}M$ form a rank n sub-bundle $T^{0,1}M$ of the complexified tangent bundle $\mathbb{C}TM$ (of rank $2n+1$). Sections of $T^{0,1}M$ are called CR vector fields.

The following properties of $T^{0,1}M$ are fundamental:

(P1) $T_p^{1,0}M \cap T_p^{0,1}M = \{0\}$;

(P2) $[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$; i.e., if X, Y are CR vector fields, then the commutator $[X, Y]$ is a CR vector field.

Note: (P1) $\implies$ $T_M^{1,0} \oplus T_p^{0,1}M$ is a complex hyperplane in $\mathbb{C}T_pM$. $\implies$ $\mathbb{C}T_pM = T_p^{1,0}M \oplus T_p^{0,1}M \oplus \mathbb{C}\langle X_p\rangle$, if $X_p \in \mathbb{C}T_pM \setminus T_M^{1,0} \oplus T_p^{0,1}M$.

Definition. M is a CR manifold (of hypersurface type) with CR bundle $T^{0,1}M$; $CR\dim M := \dim_{\mathbb{C}} T_p^{0,1}M = n$.

The first invariant of a CR manifold; the Levi form.

- Since $T^{1,0}M \oplus T^{0,1}M$ is a Hermitian sub-bundle of corank 1 in $\mathbb{C}TM$, it can be defined by a real 1-form θ. Such θ is called a "contact form". Any other is of form $\tilde{\theta} = a\theta$, $a \neq 0$ real function.
- On Ω, the differential $d = \partial + \bar{\partial}$, where

$$\partial u := \sum_{j=1}^{n+1} \frac{\partial u}{\partial z_j} dz_j.$$

- By definition, $T_p^{1,0}M = \{X_p \in T_p^{1,0}\mathbb{C}^{n+1} \colon \langle \partial\rho, X_p \rangle = 0\}$. Since $0 = d\rho = \partial\rho + \bar{\partial}\rho$ on M,

$$\theta := i\partial\rho|_M = -i\bar{\partial}\rho|_M = \overline{i\partial\rho|_M}$$

  is real and annihilates $T^{1,0}M \oplus T^{0,1}M$. Thus, $\theta = i\partial\rho|_M$ is a contact form on M.

The Levi form.

Definition: The Levi form $\mathcal{L} = \mathcal{L}_p^\theta$ of M at p is the Hermitian form

$$\mathcal{L}(X_p, Y_p) := \frac{i}{2}\langle \theta, [X, \bar{Y}]\rangle|_p, \quad X_p, Y_p \in T_p^{1,0}M.$$

where X, Y are local sections of $T^{1,0}M$ (anti-CR vector fields) extending X_p, Y_p.

- Independent of extensions (well-defined) by Cartan's identity:

$$\langle \omega, [Z, V]\rangle = -2\langle d\omega, Z \wedge V\rangle + Z\langle \omega, V\rangle - V\langle \omega, Z\rangle.$$

$$\implies \mathcal{L}(X_p, Y_p) = -i\langle d\theta, X_p \wedge Y_p\rangle.$$

- If $\tilde{\theta} = a\theta$, then $\mathcal{L}_p^{\tilde{\theta}} = a(p)\mathcal{L}_p^\theta$ by

$$d\tilde{\theta} = a d\theta + da \wedge \theta.$$

In local coordinates: $M \subset \Omega \subset \mathbb{C}^{n+1}$.

Choose local coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ in Ω, vanishing at $p \in M$:

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w), \quad \phi(z, 0, s) = \phi(0, \zeta, s) = O\left(\|(z, s)\|^{2K}\right).$$

Note: $d\phi(0) = 0 \implies \mathbb{C}T_0M = \mathbb{C}\langle \partial/\partial z_j|_0, \partial/\partial \bar{z}_j|_0, \operatorname{Re}(\partial/\partial w|_0)\rangle$.
As a local frame for $T^{1,0}M$:

$$L_j := \frac{\partial}{\partial z_j} + \frac{2i\phi_{z_j}}{1 - i\phi_s}\frac{\partial}{\partial w}, \quad j = 1, \ldots, n.$$

With $\rho = \operatorname{Im} w - \phi$, we may choose

$$\theta = i\partial\rho|_M = \frac{1}{2}(1 - i\phi_s)dw|_M - i\sum_{j=1}^{n} \phi_{z_j} dz_j|_M.$$

We may choose $(z, s) \in \mathbb{C}^n \times \mathbb{R}$ as local chart on M:

$$(z, w) \mapsto (z, w = s + i\phi(z, \bar{z}, s)).$$

CR functions and mappings.

Definition: A function h on M is CR if $\bar{L}h = 0$ for all CR vector fields $\bar{L}$.

As a mapping $h\colon M \to \mathbb{C}$,

$$h \text{ is CR} \iff h_*(T^{0,1}M) \subset T^{0,1}\mathbb{C}.$$

$$\zeta = h(z, \bar{z}, s) \implies h_*\left(\bar{L}\right) = (\bar{L}h)\frac{\partial}{\partial \zeta} + (\bar{L}\bar{h})\frac{\partial}{\partial \bar{\zeta}}.$$

Definition: A mapping $f\colon M \to M'$ is CR if $f_*(T^{0,1}M) \subset T^{0,1}M'$.

If $M' \subset \Omega' \subset \mathbb{C}^{m'}$, then:

$$f = (f_1, \ldots, f_{m'})\colon M \to M' \text{ is CR} \iff \text{each } f_j \text{ is a CR function.}$$

Basic Example: The restriction (or boundary value) of a holomorphic function/mapping to M is CR. The converse will be addressed in Lecture III.

Invariance of Levi form under CR mapping $f\colon M \to M'$.

Pick contact forms θ, θ' on M, M'. Definition of CR $\implies$ $f^*\theta' = a\theta$. For $X_p, Y_p \in T^{1,0}_p M$,

$$
\begin{aligned}
(\mathcal{L}')^{\theta'}_{f(p)}(f_* X_p, f_* Y_p) &= \frac{i}{2}\langle \theta', [f_* X, \overline{f_* Y_p}]\rangle = -i\langle d\theta', f_* X_p \wedge \overline{f_* Y_p}\rangle \\
&= -i\langle f^* d\theta', X_p \wedge \overline{Y_p}\rangle = -i\langle d(a\theta), X_p \wedge \overline{Y_p}\rangle \\
&= \mathcal{L}^{a\theta}_p(X_p, Y_p) = a(p)\mathcal{L}^{\theta}_p(X_p, Y_p).
\end{aligned}
$$

In a local frame $L_1, \ldots, L_n$ and contact form θ, $T^{1,0}_p M \cong \mathbb{C}^n$,

$$
\mathcal{L}^{\theta}(x, y) = xEy^*, \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n,
$$

where $E = E^{\theta}_p$ is the Hermitian $n \times n$ matrix with matrix elements $E_{jk} = i/2\langle \theta, [L_j, \bar{L}_k]\rangle$. If $f_* \cong B$, $n \times n'$ matrix, then Levi form invariance:

$$
aE = BE'B^*.
$$

Levi nondegenerate CR manifolds.

Definitions. 1) M is Levi nondegenerate at p if the Levi form $\mathcal{L}_p$ is nondegenerate: $Y_p \mapsto \mathcal{L}_p(\cdot, Y_p) \in (T^{1,0}M)^*$ is injective; $\iff$ $\det E_p \neq 0$.
2) M is strictly pseudoconvex at p is $\mathcal{L}_p$ is positive definite (for some θ).

The "opposite" of Levi nondegenerate is Levi flat. M is Levi flat if the Levi form $\mathcal{L}_p = 0$ for all $p \in M$.

Proposition

M is Levi flat $\iff$ M is foliated by complex manifolds Σ_t with $\dim \Sigma_t = CR \dim M = n$.

Proof. $\mathcal{L}_p = 0 \implies \mathrm{Re}(T^{1,0}M \oplus T^{0,1}M)$ involutive. Frobenius Theorem $\implies$ M foliated by Σ_t with $T\Sigma_t = \mathrm{Re}(T^{1,0}M \oplus T^{0,1}M)$.
Newlander-Nirenberg Theorem $\implies$ Σ_t are complex manifolds. Converse is easy. See [1] for FT and NNT; will also appear in Lecture II. □

Remark. $\partial\mathbb{B}_2$ is strictly pseudoconvex, and $\partial\mathbb{D}^2$ is Levi flat (at smooth points).

Levi form in terms of a defining function ρ.

- $T_p^{1,0}M \subset T_p^{1,0}\mathbb{C}^{n+1} \cong \mathbb{C}^{n+1}$ and $x \in T_p^{1,0}M$ if

$$\sum_{j=1}^{n+1} \frac{\partial \rho}{\partial z_j}(p) x_j = 0.$$

- Choose $\theta = i\partial\rho|_M$.

$$\mathcal{L}^\theta(X, Y) = i\langle d\theta, X \wedge Y\rangle = \langle \partial\bar{\partial}\rho, X \wedge Y\rangle$$
$$= \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \bar{z}_k}(p) x_j \bar{x}_k.$$

- Thus, $\mathcal{L}^\theta \sim (n+1) \times (n+1)$ matrix $F = (\partial^2\rho/\partial z_k \partial \bar{z}_k)$, restricted to n-dimensional subspace $T_p^{1,0}M \subset \mathbb{C}^{n+1}$.

Fefferman's complex Monge-Ampére Operator.

Consider the complex Monge-Ampére type operator in $\mathbb{C}^{n+1}$:

$$J(u) := (-1)^{n+1} \det \begin{pmatrix} u & u_{\bar{z}} \\ u_z & u_{z\bar{z}} \end{pmatrix}.$$

Proposition

Let $M \subset \Omega \subset \mathbb{C}^{n+1}$ be defined by $\rho = 0$. Then, M is Levi nondegenerate at $p \in M \iff J(\rho)|_p \neq 0$.

Proof. Let $p \in M$, and F denote $(n+2) \times (n+2)$ matrix such that $J(\rho) = (-1)^{n+1} \det F$. Pick $\tilde{x} = (c, x) \in \mathbb{C} \times \mathbb{C}^{n+1}$. Then, $\tilde{x}F = 0 \iff x\rho_z = 0$ and $c\rho_{\bar{z}} + x\rho_{z\bar{z}} = 0$. Note that there is $c \in \mathbb{C}$ such that $c\rho_{\bar{z}} + x\rho_{z\bar{z}} = 0 \iff x\rho_{z\bar{z}}y^* = 0$ for all $\rho_{\bar{z}}y^* = 0$. □

Fefferman's defining equations.

Theorem (Fefferman, '76 [3])

Let $M \subset \Omega \subset \mathbb{C}^{n+1}$ *be strictly pseudoconvex, defined by* $\rho = 0$*. Then, there is a unique, mod* $O(\rho^{n+3})$*, defining function* r *for* M *such that* $J(r) = 1 + O(\rho^{n+2})$.

Remark: Such r is called a Fefferman defining function for M. Useful for studying invariants of strictly pseudoconvex domains, notably their Bergman kernels.

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