COMPLEX DYNAMICAL SYSTEMS

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1. Introduction to Fatou components

The goal in this course is to present some recent results in higher dimensional holomorphic dynamics. In order to be able to appreciate these results, one needs to have seen a certain amount of one-dimensional dynamics. Hence we will start by looking at the one-dimensional setting. We will only present detailed proofs in the direction that we will take in higher dimensions.

Let X be a complex manifold, $f : X \to X$ holomorphic. What can we say about the *orbits* $z_0, f(z_0), f^2(z_0), \ldots$? A central question for us will be: how do the orbits vary when the initial value z_0 is perturbed.

Example 1. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be given by $f(z) = z^d$, for some $d \geq 2$. The point $0 = f(0)$ is fixed, and the orbits of all nearby points converge to the origin, since $|f'(0)| = 0 < 1$. The situation is *stable*. The other fixed point is $1 = f(1)$. However, as $f'(1) = d > 1$, the orbit of any nearby initial value will escape a small neighborhood of 1. The situation is *unstable*.

Let us be more precise about stability. For the next definition we assume that there exists a metric $d(\cdot, \cdot)$ on X.

Definition 2. An initial value z_0 lies in the Fatou set F_f if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d(w_0, z_0) < \delta$ implies that $d(w_n, z_n) < \epsilon$ for all $n \in \mathbb{N}$.

In other words, the family $\{f^n\}$ is equicontinuous in the point z_0. For holomorphic maps this implies equicontinuity in a neighborhood U of z_0, which is equivalent to the condition that every sequence $(f^{n_j}|_U)$ has a convergent subsequence. Here the topology used is the compact-open topology, and convergence means uniform convergence on compact subsets.

Definition 3. The Julia set is the complement of the Fatou set, $J_f := X \setminus F_f$.

The following questions are central in holomorphic dynamics.

(1) Describe the behavior of orbits on J_f.
(2) Describe the behavior of orbits on F_f.
(3) How do natural invariant sets such as J_f and F_f vary when the map f is perturbed.

It turns out that in order to understand the third question a good understanding of the previous two questions is needed. This is for obvious reasons, if you do not understand the sets J and F then it is unlikely that you can determine how these sets will vary as the parameters vary. It turns out that understanding question (3) can also help to understand the answers to questions (1) and (2), although this is more subtle. A classical example can be found in the works of Shishikura

Date: October 2015, SCTP Trieste.

and Buff-Cheritat, who proved that there exist polynomials whose Julia sets have Hausdorff measure 2, resp. positive Lebesque measure. A more recent example is the construction of wandering Fatou components for polynomials in two complex variables.

The first two questions are closely related, and one rarely studies the Fatou set without using properties of the Julia set. In this course we will mostly focus on the second question.

We will mostly consider rational maps. Of course there are many other interesting dynamical systems, also in the holomorphic category. In fact transcendental dynamics has received significant attention in the literature. However, it is important to note that it is almost impossible to prove anything of substance about very large classes of maps, such as *all* entire maps, or *all* holomorphic automorphisms of $\mathbb{C}^2$. These classes are too large to study their dynamics (unless one wants to show that specific behavior can occur for some maps, then having a large class of maps is of course an advantage).

In the one-dimensional setting, the Riemann surface $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a particularly pleasant space to study the dynamics. The dynamical behavior of holomorphic endomorphisms is much richer there than in any other compact Riemann surface.

Homework 1. Prove that all holomorphic endomorphisms of $\hat{\mathbb{C}}$ are rational.

Homework 2. Describe the behavior of affine maps $z \mapsto az + b$. What are the Fatou and Julia sets?

The family of quadratic polynomials is one of the most studied dynamical systems. Note that we can *conjugate* any quadratic polynomial to a polynomial of the form $f(z) = z^2 + c$. Another popular form is $z \mapsto kz(1-z)$, which is called a logistic map. Of course one can easily change coordinates from one form to another. The advantage of the form $f(z) = z^2 + c$ is that it stresses the importance of the *critical point* $z = 0$, the point where the derivative vanishes, and the critical value c. We will see many examples this week that highlight the important role of the critical point in complex dynamical systems.

Homework 3. Prove that a rational function of degree d has $2d-2$ critical points, counting multiplicity.

For a polynomial of degree d the point ∞ always is a critical point of order $d-1$, meaning that the derivative vanishes at ∞ with order $d-1$. Hence there are exactly $d-1$ critical points in the complex plane, counting with multiplicity as always.

Let us look at a particular example of a quadratic polynomial. We will not choose the parameter $c = 0$, because besides being rather boring, the polynomial $f(z) = z^2$ is also quite different from any other polynomial $z^2 + c$. It will be important for us to know what happens to the orbit of the critical point 0. Let us therefore take an easy example where 0 is a periodic point. The period 1 case gives the map we do not want, so let us take a higher order, for example order 3. Now we need to solve the equation

$$f_c^3(0) = 0,$$

which is a (real) degree 4 polynomial in c, hence has 4 solutions. There are two real solutions, one of which is 0. There is also a pair of complex solutions. Let us pick one of these complex solutions, simply because the picture is nicer (at least I think so). See Figure 1.

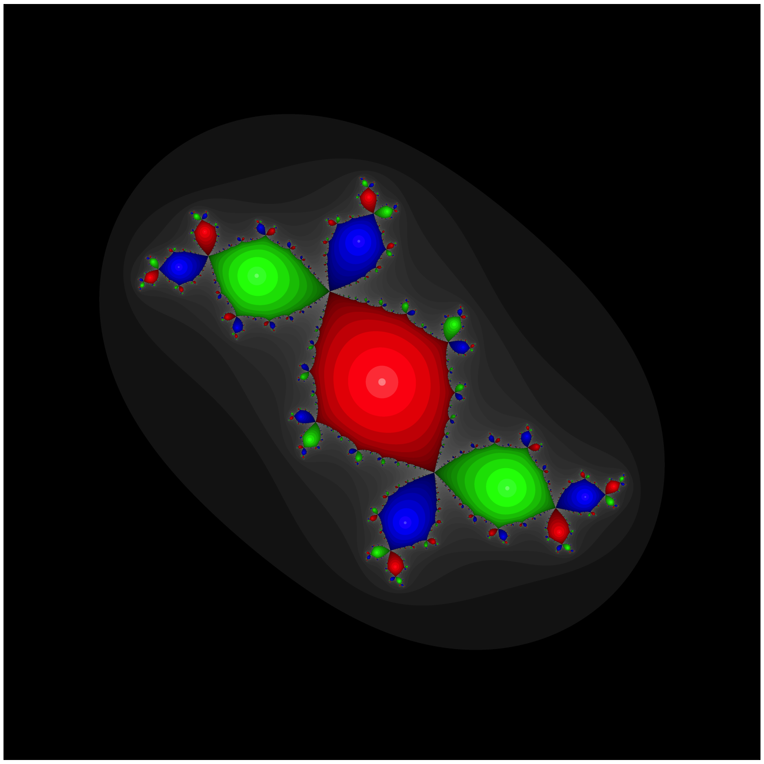

FIGURE 1. $z \mapsto z^2 + c$

What do we see in this picture? The region depicted in black (and dark grey) can be regarded as the *basin of attraction* of the fixed point at infinity,

$$I_\infty = \{z \in \mathbb{C} \mid f^n(z) \to \infty\}.$$

Note that for any polynomial there exists a constant $R > 0$ such that if $|z| \geq R$ then $|f(z)| \geq 2|z|$. It follows that if an orbit at any point leaves the disk of radius R, then it must converge to infinity. The constant R is often called the escape radius, although of course it is not unique. Drawing the black (grey) region by computer is easy. Choose coordinates for each pixel and iterate until the orbits leaves the disk $\Delta(R)$. The grey scale shows how many steps it takes to leave this disk. If the orbit has not left the disk after a large number of iterates then we do not color it grey. For a sufficiently large number of iterates this gives a very good approximation of I_∞.

The complement of I_∞ is denoted by K,

$$K := \{z \in \mathbb{C} \mid \{f^n(z)\} \text{ is bounded}\}.$$

Since I_∞ is open, the set K is closed. Both sets are completely invariant under f.

Homework 4. Prove that $J = \partial K$, and $F = K^\circ \cup \left(\hat{\mathbb{C}} \setminus K\right)$.

In the picture it appears that K is depicted in green, red and blue. This is almost true, but not quite. Note that $0 \mapsto c \mapsto f(c) \mapsto 0$. The derivative of f^3 at each of these three points is 0.

Lemma 4. *Let $f(p) = p$ and suppose that $|f'(p)| < 1$. Then there is a neighborhood $U(p)$ such that the orbit of any point in U converges to p. Conversely, suppose that $U \subset \mathbb{C}$ is open and that $f(U)$ is a relatively compact subset of U. Then there is a point $p \in U$ with $|f'(p)| < 1$, and the orbit of any point in U converges to p.*

Homework 5. Prove the above lemma.

The points $0, c$ and $f(c)$ are attracting fixed points for f^3, hence $\{0, c, f(c)\}$ is an attracting periodic cycle for f. The attracting basin for f^3 of the point 0 is depicted in red, for c in blue, and for $f(c)$ in green. Together they form the basin A of the attracting periodic cycle of f. Note that A is again completely invariant, i.e. $f^{-1}(A) = A$.

Let us write A_0 for the connected component of A that contains 0, and write $A_1 = f(A_0)$ and $A_2 = f(A_1)$.

Homework 6. Prove that the components A_0, A_1 and A_2 are distinct, and thus disjoint.

The maps $f : A_1 \mapsto A_2$ and $f : A_2 \mapsto A_0$ are locally injective (since the critical point is not contained in A_0). By the Maximum Principle these components are simply connected. It follows that the restriction of f to either A_1 or A_2 is injective. The map $f : A_0 \to A_1$ is not injective but $2 : 1$, except that c has only 1 inverse image.

Since the polynomial f has degree 2, it follows from the above discussion that the components A_2 and A_0 each must have another pre-image. Each of these new components must also have two pre-images, and so on. It follows that the set A consists of countably many distinct connected components. The combinatorial data of these connected components gives us a complete description of all possible orbits of f.

Claim 1. $K^\circ = A$.

Claim 2. $\partial A = J = \partial$"red set" $= \partial$"blue set" $= \partial$"green set".

Claim 3. Define the sets A_{-n} for $n \in \mathbb{N}$ recursively so that $f(A_{-n}) = A_{-n+1}$, and $A_{-3} \neq A_0$. Then $\text{diam}(A_{-n}) \to 0$.

Before we prove these three claims, it is worth noting that while we are looking at a very specific polynomial, these properties hold in much greater generality. Indeed, they follow from some very general results.

It is immediately clear that $\text{Area}(A_{-n}) \to 0$, since the sets A_{-n} are pairwise disjoint and are all contained in $\Delta(R)$. Claim 3 is of course strictly stronger. How do we prove this claim? Note again that each component A_{-n} is simply connected and does not contain critical points. Therefore the map $f : A_{-n} \to A_{-n+1}$ is injective, and thus $f^n : A_{-n} \to A_0$ is also univalent. Therefore we can consider the inverse branches $f^{-n} : A_0 \to A_{-n}$, which are of course also univalent, and bounded since the images are contained in $\Delta(R)$. Hence these inverse branches form a normal family, and any sequence of these inverse branches must have a convergent subsequence, uniformly on compact subsets of A_0. Since all the images A_{-n} are pairwise disjoint, it follows that the image of any limit map $h = \lim f^{-n_j}$ must be a single point. After all, h must be holomorphic, and $h(A_0)$ cannot have any interior points. If it did, then some (open) set A_{-n} would intersect an interior point of $h(A_0)$, in which case that point could never be approximated by points in other sets A_{-m}.

Now we wish to conclude from the fact that any limit set $h(A_0)$ consists of a single point, that the diameter of the sets A_{-n} must converge to zero. We could conclude this immediately if the convergence was uniform. However, the convergence is only uniform, it is only uniform on compact subsets. This is not sufficient to conclude that the diameters of A_{-n} converge to zero. It is possible that the sets A_{-n} become very thin but long. How can we prove that this does not happen?

Definition 5. We say that a polynomial f is *hyperbolic* if some iterate f^N acts expandingly on the Julia set J.

Expanding means that $|f'(z)| > 1$ for any $z \in J$. By compactness of J it follows that there exists a $\mu > 1$ such that $|f'(z)| > \mu$ for any $z \in J$. An equivalent definition of hyperbolicity is that there exists a metric on J with respect to which f is uniformly expanding.

A clear example of a hyperbolic polynomial is the map $z \mapsto z^2$. The Julia set of this map is the unit circle, where the derivative has constant norm 1. It turns out that the polynomial f_c that we have been considering is also hyperbolic. Therefore it is also uniformly expanding in a given neighborhood of the Julia set. If the sets A_n get sufficiently thin then they must be contained in this neighborhood of J. But then the branches of f^{-1} that we are considering are uniform contractions on the closure of the sets A_{-n}, from which it follows that the diameters of A_{-n} converge to zero.

Looking back we notice that we actually did not need to go through the normality argument. The fact that the area convergence to zero is enough in combination with the univalence and the hyperbolicity. These two ingredients, the normality argument and the hyperbolicity, will however play a crucial role in what will come later in the course.

2. A short section on Equilibrium measures

Take any point $z \in \hat{\mathbb{C}} \setminus \{\infty\}$, and define the probability measures

$$\mu_n = \frac{1}{2^n} \sum_{f^n(w)=z} \delta_w.$$

As usual we need to sum over all w counting multiplicities.

One can show that the measures μ_n converge weakly to a probabily measure μ which is independent of the chosen point z. The measure μ is called the equilibrium measure of f. It if invariant under f and $\mathrm{supp}(\mu) = J$. Many other natural constructions lead to the same measure μ, and this measure plays an important role in our understanding of one-dimensional complex dynamical systems. The above construction works for any rational function, except for the exceptional polynomials conjugate to $z \mapsto z^d$, in which case one should not start with the two completely invariant critical points.

The construction of the equilibrium measure is due to Brolin (for polynomials, 1965), Lyubich (1982, for rational functions) and Mañé. It has been successfully generalized to higher dimensional maps, starting with the works of Bedford-Smillie (for polynomial automorphisms), Sibony-Fornæss (for holomorphic endomorphisms of projective space), and later for many more maps, for an important part due to Sibony.

3. Back to the Fatou set

Our earlier claim 2, stating that the boundary of each of the red, green and blue sets equals J, follows from the above comment that $\mathrm{supp}(\mu) = J$, but there is a much easier way to prove the claim. Let $z \in J$, and let U be a neighborhood of z. Then U must contain both red, blue and green points, otherwise the family of iterates defined on U avoids at least 2 points of $\mathbb{C}$ and by Montel's theorem is normal. But this would contradict the definition of J.

Our last claim, which was in fact the first claim, states that $A = K^\circ$, which is equivalent to saying that the Fatou set has no other connected components. Connected components of the Fatou set are called Fatou components, and for rational functions they are very well understood. The proof of the last claim will follow from the following two theorems.

Theorem 6 (Fatou, Siegel, Herman, Sullivan)**.** *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be rational of degree at least 2, and let U be a Fatou component. Then U is either periodic or pre-periodic. If U is invariant then U must either be*

- Attracting basin *the immediate basin of an attracting fixed point p, i.e. $f(p) = p$ and $|f'(p)| < 1$.*
- Parabolic basin *an immediate basin of a parabolic fixed point p, i.e. $f(p) = p$ and $f'(p) = 1$.*
- Siegel disk*: equivalent to a disk on which f acts as an irrational rotation.*
- Herman ring*: equivalent to an annulus on which f acts as an irrational rotation.*

The classification is due to Fatou, but he was not able to decide whether the third and fourth cases could exist. The existence of Siegel disks was shown by Siegel in the 1940's, the existence of Herman rings was shown by Herman in 1979. In both cases there exists a conformal map $\varphi : U \mapsto \Sigma$ such that $\phi(f(z)) = \lambda \cdot \phi(z)$, where $\lambda = e^{2\pi i\theta}$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Here Σ is either the unit disk (in the Siegel case) or an annulus (in the Herman case). Note that Herman rings cannot exist for polynomials. It was shown by Sullivan in 1985 that every Fatou component is either periodic or pre-periodic. A Fatou component that is not periodic or pre-periodic is called *wandering*. For this reason one often refers to Sulivan's No Wandering Domains Theorem.

Theorem 7 (Fatou components and critical points)**.** *The immediate basin of an attracting fixed point or attracting parabolic point must contain a critical point. The boundary of a Siegel disk must be contained in the closure of a critical orbit.*

Since for our specific polynomial the critical orbit is a finite (and thus compact) subset of $A_0 \cup A_1 \cup A_2$ it follows that there can be no invariant Siegel disks, parabolic basins or other attracting basins. This not only holds for f but also for f^n for any n. Hence there can be no other periodic Fatou components, and it follows from Sullivan's No Wandering Domains Theorem that there exist no other Fatou components, hence $A = K^\circ$.