CR Geometry, Mappings into Spheres, and Sums-Of-Squares
Lecture II

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Outline - Lecture II

Formally integrable structures.

Let $M = M^m$ be a manifold of dimension m and class C^κ; $\kappa = \infty, \omega$.

Definition. A formally integrable structure of rank n on M is a subbundle $\mathcal{V} \subset \mathbb{C}TM$ (of rank n) such that:

$$[X, Y] \in \Gamma(U, \mathcal{V}), \quad \forall X, Y \in \Gamma(U, \mathcal{V}), \; U \subset M. \tag{1}$$

Remarks. 1) We use the notation $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ to abbreviate (1).
2) Any rank n subbundle $\mathcal{V} \subset \mathbb{C}TM$ is defined locally by $m - n$ linearly independent 1-forms $\omega^1, \ldots, \omega^{m-n}$; By Cartan's identity, $\mathcal{V}$ is formally integrable $\iff$ $d\omega^j$ all vanish on $\Lambda^2\mathcal{V}$.

Definition. A formally integrable structure $\mathcal{V}$ of rank n on $M = M^m$ is (locally) integrable if every $p \in M$ has a neighborhood U and $Z^1, \ldots Z^{m-n} \in C^\kappa(U, \mathbb{C})$ such that $\omega^j = dZ^j$, , i.e., for $j = 1, \ldots, m - n$,

$$XZ^j = \langle dZ^j, X \rangle = 0, \quad \forall X \in \Gamma(U, \mathcal{V}); \quad dZ^1 \wedge \ldots \wedge dZ^{m-n} \neq 0.$$

$Z = (Z^1, \ldots, Z^{m-n})$ is called a system of solutions (or first integrals).

The Frobenius Theorems. See e.g. Treves [8] for proofs.

Real Frobenius Theorem

Let $M = M^m$ be a real C^κ manifold with rank n subbundle $E \subset TM$ such that $[E, E] \subset E$. Then, every $p \in M$ has a neighborhood U and $F^1, \ldots, F^{m-n} \in C^\kappa(U, \mathbb{R})$ such that $E^\perp$ is spanned by $dF^1, \ldots, dF^{m-n}$.

Holomorphic Frobenius Theorem

Let $\Omega = \Omega^m$ be a complex manifold with rank n holomorphic subbundle $\mathcal{V} \subset T^{1,0}\Omega$ such that $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$. Then, every $p \in \Omega$ has a neighborhood U and $Z^1, \ldots, Z^{m-n} \in \mathcal{O}(U)$ such that $\mathcal{V}^\perp$ is spanned by $dF^1, \ldots, dF^{m-n}$.

Remarks: 1) Frobenius + IFT $\implies \exists$ local charts $x = (u, v)$ such that M is foliated by submanifolds $\Sigma_q := \{(u, v) \colon v = q\}$ and E coincides with $T\Sigma_q$. Similarly for holomorphic Frobenius.
2) Neither Frobenius nor IFT applies to formally integrable structures in general!

Real-analytic formally integrable structures.

Theorem

A real-analytic formally integrable structure $(M, \mathcal{V})$ is locally integrable.

Proof. Let $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ be a local chart near $p \cong 0$ in $M \cong U \subset \mathbb{R}^m$, and $L_1, \ldots, L_n$ a local basis for sections of $\mathcal{V} \subset \mathbb{C}T\mathbb{R}^m$:

$$L_j = \sum_{k=1}^{m} a_{jk}(x) \frac{\partial}{\partial x_k}, \quad a_{jk} \in C^{\omega}.$$

Complexify! Consider $x = (x_1, \ldots, x_m) \in \mathbb{C}^m$ as complex coordinates, the $a_{jk}(x)$ become holomorphic, and L_j holomorphic $(1, 0)$ vector fields, spanning a holomorphic complexified subbundle $\mathcal{V}^{\mathbb{C}} \subset T^{1,0}\mathbb{C}^m$. Frobenius integrability $[\mathcal{V}^{\mathbb{C}}, \mathcal{V}^{\mathbb{C}}] \subset \mathcal{V}^{\mathbb{C}}$ follows from $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ in $\mathbb{R}^m$. By holomorphic Frobenius, there are holomorphic $Z^1, \ldots Z^n$ such that $\mathcal{V}^{\mathbb{C}}$ is given by $dZ^j = 0$. The functions $Z^j(x)$, for x real, provide the desired system of solutions $Z = (Z^1, \ldots, Z^{m-n})$ in $U \subset \mathbb{R}^m$. □

Representation of locally integrable structures.

Let $Z = (Z^1, \ldots, Z^{m-n})$ be a system of solutions in $U \subset M = M^m$ of a rank n, integrable structure $\mathcal{V} \subset \mathbb{C}TM$. Consider $Z\colon U \to \mathbb{C}^{m-n}$. Recall chain rule:

$$Z_*(X_p) = \sum_{k=1}^{m-n} \left(\langle dZ^k, X_p \rangle \frac{\partial}{\partial Z^k} + \langle d\bar{Z}^k, X_p \rangle \frac{\partial}{\partial \bar{Z}^k} \right), \quad X_p \in \mathbb{C}T_pM.$$

$\implies \ker Z_* = \mathcal{V} \cap \overline{\mathcal{V}}$. The Rank Theorem ("IFT Plus") $\implies$

Proposition

If $\mathcal{K} := \mathcal{V} \cap \overline{\mathcal{V}}$ is a subbundle of rank k, then $Z(U) \subset \mathbb{C}^{m-n}$ is an immersed real submanifold of dimension $m-k$. The map $Z\colon U \to Z(U)$ is a submersion such that $Z_*\mathcal{V} \subset T^{0,1}\mathbb{C}^{m-n}$, whose fibers $Z^{-1}(q) \subset U$ are submanifolds of dimension k with $TZ^{-1}(q) = \mathcal{V} \cap \mathcal{V}$.

CR structures.

Definitions. 1) A formally integrable structure $\mathcal{V}$ on M is CR if

$$\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}.$$

2) A mapping $f\colon (M, \mathcal{V}) \to (M', \mathcal{V}')$ is CR if $f_*(\mathcal{V}) \subset \mathcal{V}'$. A CR function is a CR mapping $f\colon (M, \mathcal{V}) \to (\mathbb{C}, T^{0,1}\mathbb{C}) \iff \bar{L}f = 0$ for all CR vector fields $\bar{L}$.

- If $\dim M = m$, $\operatorname{rank} \mathcal{V} = n$, then $m = 2n + d$ for some $d \geq 0$;
- CR dim M=n; CR codim $M = d$.
- A locally integrable CR manifold $(M, \mathcal{V})$ can be locally embedded in $\mathbb{C}^{n+d}$ by the CR mappings $Z\colon (U, \mathcal{V}) \to (\mathbb{C}^{n+d}, T^{0,1}\mathbb{C}^{n+d})$, $U \subset M$.
- A real-analytic CR manifold is locally integrable, and hence locally embeddable in $\mathbb{C}^{n+d}$.

CR functions and mappings for embedded CR manifolds.

Assume that $(M, \mathcal{V})$ is embedded as a real submanifold in $\Omega \subset \mathbb{C}^m$; i.e.,

$$M \subset \mathbb{C}^m, \quad \mathcal{V} = \mathbb{C}TM \cap T^{0,1}\Omega.$$

If $M' \subset \Omega' \subset \mathbb{C}^{m'}$, then:

$$f = (f_1, \ldots, f_{m'})\colon M \to M' \text{ is CR } \iff \text{ each } f_j \text{ is a CR function.}$$

Basic Example: The restriction (or boundary value) of a holomorphic function/mapping to M is CR. The converse will be addressed in Lecture III.

Almost complex structures and complex manifolds.

- An almost complex structure on $M = M^{2d}$ is a rank $n = d$ subbundle $\mathcal{V}$ such that $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$. $\implies \mathbb{C}TM = \mathcal{V} \oplus \overline{\mathcal{V}}$.
- If $(M, \mathcal{V})$ is formally integrable, then it is a CR structure of CR dim M=0.
- $(M, \mathcal{V})$ is locally integrable $\iff$ M is a complex manifold with $\mathcal{V} = T^{0,1}M$: local charts are given by local systems of solutions $Z\colon U \to \mathbb{C}^d$.

Newlander-Nirenberg Theorem; [6], [2]

A formally integrable almost complex structure $(M, \mathcal{V})$ is locally integrable.

Examples of CR manifolds with positive codimension.

Example 1. Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface; i.e., locally,

$$M\colon \rho(z,\bar{z}) = 0, \quad \rho \in C^\kappa(U,\mathbb{R}),\ d\rho|_M \neq 0.$$

$M = M^{2n+1}$ is a CR manifold with $\mathcal{V} = T^{0,1}M := \mathbb{C}TM \cap T^{0,1}\mathbb{C}^{n+1}$; CR dim $M = n$, CR codim $M = 1$.

Definition. CR manifolds with CR codim $M = 1$ are of hypersurface type.

Example 2. Let $M \subset \mathbb{C}^N$ be a real submanifold of codimension k; i.e.,

$$M\colon \rho(z,\bar{z}) = 0, \quad \rho \in C^\kappa(U,\mathbb{R}^k),\ d\rho_1 \wedge \ldots \wedge d\rho_k|_M \neq 0.$$

When $k \geq 2$, the spaces $T^{0,1}_pM := \mathbb{C}T_pM \cap T^{0,1}_p\mathbb{C}^N$ may not have constant dimension. But, if

$$\mathrm{rank}_{\mathbb{C}}\{\bar\partial\rho_1,\ldots,\bar\partial\rho_k\} = \text{constant } = d, \tag{2}$$

then M is a CR manifold with $\mathcal{V} := T^{0,1}M$; dim $M = 2N - k$, CR dim $M = N - d$, CR codim $M = 2d - k$. If $d = k$, then M is generic.

A non-CR submanifold, totally real manifolds, and a topological fact.

Example 3. Consider a 2-sphere S^2 in $\mathbb{C}^2$, e.g.,

$$S^2\colon\ \rho_1 := |z|^2 + |w|^2 - 1 = 0,\ \ \rho_2 := \operatorname{Im} w = 0.$$

$\implies \bar\partial\rho_1 = z d\bar z + w d\bar w$, $\bar\partial\rho_2 = -i d\bar w/2$; $\implies T^{0,1}_p S^2 = \{0\}$, except at the two points $p = \pm(0,1)$, where $T^{0,1}_p S^2$ equals a complex line spanned by $\partial/\partial\bar z$. Thus, S^2 is a real submanifold of codimension $k = 2$ in $\mathbb{C}^2$, but it is not CR.

Definition: A real submanifold $M \subset \Omega \subset \mathbb{C}^m$ is totally real if the induced CR structure is trivial, $\mathcal{V} := T^{0,1}M = \{0\}$.

Theorem (Wells [9])

If $M = M^2 \subset \mathbb{C}^2$ is a compact, totally real surface, then M is a torus.

Levi form(s) of a CR manifold ($M = M^{2n+d}, \mathcal{V} = \mathcal{V}^n$).

$H_{\mathbb{C}} := \mathcal{V} \oplus \overline{\mathcal{V}}$ is a Hermitian subbundle of rank $2n$ and corank d in $\mathbb{C}TM$. The rank d (characteristic) bundle $H_{\mathbb{C}}^{\perp} \subset \mathbb{C}T^*M$ can then be spanned, locally, by d linearly independent, *real* 1-forms $\eta^1, \ldots \eta^d$. We set $H := \operatorname{Re} H_{\mathbb{C}} \subset TM$, and then $\eta^1, \ldots \eta^d$ span $H^{\perp} \subset T^*M$. If

$$\theta = a_j \eta^j := \sum_{j=1}^{d} a_j \eta^j \quad \text{(summation convention)}$$

is a characteristic form (section of $H^{\perp}$), then the Levi form at θ is defined, for $X_p, Y_p \in \bar{\mathcal{V}}_p$, by

$$\mathcal{L}(X_p, Y_p) = \mathcal{L}_p^{\theta}(X_p, Y_p) := \frac{i}{2} \langle \theta, [X, \bar{Y}] \rangle = -i \langle d\theta, X_p \wedge \bar{Y}_p \rangle.$$

(Invariantly: $\mathcal{L}(X_p, Y_p) = \pi([X, \bar{Y}])$, $\pi \colon \mathbb{C}T_p \to \mathbb{C}T_p / T_p^{1,0} \oplus T_p^{0,1} \cong \mathbb{C}^d$.)

Invariance of Levi form under CR mappings $f\colon M \to M'$.

The Levi form $\mathcal{L}$ can be viewed as a tensor in $\overline{\mathcal{V}}^* \otimes \mathcal{V}^* \otimes (H^\perp)^*$. Pick local bases $L_1, \ldots, L_n$ and $\eta^1, \ldots, \eta^d$ for $\overline{\mathcal{V}}$ and H; $\overline{\mathcal{V}}_p \cong \mathbb{C}^n$ and $H_p^\perp \cong \mathbb{R}^d$. Then $\mathcal{L} = ({h_{\alpha\bar\beta}}^j) \in \overline{\mathcal{V}}^* \otimes \mathcal{V}^* \otimes (H^\perp)^*$,

$$ {h_{\alpha\bar\beta}}^j := \mathcal{L}_p^{\eta^j}(L_\alpha, L_\beta). $$

If we change bases $L_\alpha = {b_\alpha}^\gamma \tilde{L}_\gamma$, $\eta^j = {a^j}_k \tilde{\eta}^k$, then

$$ {h_{\alpha\bar\beta}}^j = {\tilde{h}_{\gamma\bar\mu}}{}^k {b_\alpha}^\gamma {b_{\bar\beta}}^{\bar\mu} {a^j}_k, \quad {b_{\bar\beta}}^{\bar\mu} := \overline{{b_\beta}^\mu}. $$

Invariance of Levi form. If $f_* L_\alpha = {b_\alpha}^{\beta'} L'_{\beta'}$, $f^* \eta'_{j'} = {a^{j'}}_k \eta^k$, then

$$ {a^{j'}}_j {h_{\alpha\bar\beta}}^j = {h'_{\alpha'\bar\beta'}}{}^{j'} {b_\alpha}^{\alpha'} {b_{\bar\beta}}^{\bar\beta'}. $$

Proof: Same computation as in Lecture I... But also on next slide in the hypersurface case, since we did not get to it last lecture. □

Invariance under $f\colon M \to M'$; hypersurface case.

Pick contact forms θ, θ' on M, M'. Definition of CR $\implies$ $f^*\theta' = a\theta$. For $X_p, Y_p \in T_p^{1,0}M$,

$$(\mathcal{L}')^{\theta'}_{f(p)}(f_*X_p, f_*Y_p) = \frac{i}{2}\langle \theta', [f_*X, \overline{f_*Y_p}]\rangle = -i\langle d\theta', f_*X_p \wedge \overline{f_*Y_p}\rangle$$
$$= -i\langle f^*d\theta', X_p \wedge \overline{Y_p}\rangle = -i\langle d(a\theta), X_p \wedge \overline{Y_p}\rangle$$
$$= \mathcal{L}^{a\theta}_p(X_p, Y_p) = a(p)\mathcal{L}^{\theta}_p(X_p, Y_p).$$

In a local frame $L_1, \ldots, L_n$ and contact form θ, $T_p^{1,0}M \cong \mathbb{C}^n$,

$$\mathcal{L}^\theta(x, y) = xEy^*, \quad x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n,$$

where $E = E_p^\theta$ is the Hermitian $n \times n$ matrix with matrix elements $E_{jk} = i/2\langle \theta, [L_j, \bar{L}_k]\rangle$. If $f_* \cong B$, $n \times n'$ matrix, then Levi form invariance:

$$aE = BE'B^*.$$

Hans Lewy's Example and nonintegrable CR structures.

Consider the Lewy operator (vector field)

$$\bar{L} = \frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial s}, \quad (z, s) \in \mathbb{C} \times \mathbb{R}.$$

Note that $\bar{L}$ defines the CR structure on the Lewy hypersurface $M\colon\ \operatorname{Im} w = |z|^2$ in $\mathbb{C}^2$, in the coordinates (z, s).

Theorem (Lewy [5])

There exist (many) $v \in C^\infty$ near 0 such that $\bar{L}u = v$ has no C^1 solutions near 0.

Remark. By the classical Cauchy-Kowalevski Theorem, for every $v \in C^\omega$, there are C^ω solutions u.
A modification of the construction in the proof of Lewy's Theorem, yield examples of nonintegrable CR structures. The first example was given by Nirenberg [7]. The reader is referred to [3] for a readable account of these constructions.

Nirenberg's Example of a nonintegrable CR structure.

Theorem (Nirenberg [7])

There exist (many) $v \in C^\infty$ near 0, vanishing at 0 to infinite order, such that

$$\bar{L}'u = 0, \quad \bar{L}' = \bar{L} + iv\frac{\partial}{\partial s}$$

has no C^1 solutions near 0 other than the constants.

- $\bar{L}'$ defines a C^∞ CR structure $\mathcal{V}'$ on $M' = \mathbb{C} \times \mathbb{R}$. (Note that formal integrability is *automatic* for rank 1 bundles.)
- Local integrability near 0 would require two solutions $u = Z^1$, $u = Z^2$ to $\bar{L}'u = 0$ with $dZ^1 \wedge dZ^2 \neq 0$, which is of course impossible by Nirenberg's Theorem. $\implies$ the CR manifold M' is not locally integrable at 0.
- The CR structure of M' agrees up to infinite order with that of the Lewy hypersurface M. In particular, M' is strictly pseudoconvex near 0.

The local integrability problem for strictly pseudoconvex CR manifolds.

Theorem (Kuranishi $\dim M \geq 9$, [4]; Akahori $\dim M = 7$, [1])

Let $(M, \mathcal{V})$ be a C^∞ CR manifold of hypersurface type (CR $\operatorname{codim} M = 1$). Assume that M is strictly pseudoconvex and $\dim M \geq 7$ (CR $\dim M \geq 3$). Then, $(M, \mathcal{V})$ is locally integrable.

Remark. As mentioned, Nirenberg showed that the conclusion does not hold when $\dim M = 3$.

Problem. What about $\dim M = 5$?

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