COMPLEX DYNAMICAL SYSTEMS

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1. Lecture 1: Fatou components in one complex variable

1.1. **Introduction to Fatou components.** The goal in this course is to present some recent results in higher dimensional holomorphic dynamics. In order to be able to appreciate these results, one needs to have seen a certain amount of one-dimensional dynamics. Hence we will start by looking at the one-dimensional setting. We will only present detailed proofs in the direction that we will take in higher dimensions.

Let X be a complex manifold, $f : X \to X$ holomorphic. What can we say about the *orbits* $z_0, f(z_0), f^2(z_0), \ldots$? A central question for us will be: how do the orbits vary when the initial value z_0 is perturbed.

Example 1. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be given by $f(z) = z^d$, for some $d \geq 2$. The point $0 = f(0)$ is fixed, and the orbits of all nearby points converge to the origin, since $|f'(0)| = 0 < 1$. The situation is *stable*. The other fixed point is $1 = f(1)$. However, as $f'(1) = d > 1$, the orbit of any nearby initial value will escape a small neighborhood of 1. The situation is *unstable*.

Let us be more precise about stability. For the next definition we assume that there exists a metric $d(\cdot, \cdot)$ on X.

Definition 2. An initial value z_0 lies in the Fatou set F_f if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d(w_0, z_0) < \delta$ implies that $d(w_n, z_n) < \epsilon$ for all $n \in \mathbb{N}$.

In other words, the family $\{f^n\}$ is equicontinuous in the point z_0. For holomorphic maps this implies equicontinuity in a neighborhood U of z_0, which is equivalent to the condition that every sequence $(f^{n_j}|_U)$ has a convergent subsequence. Here the topology used is the compact-open topology, and convergence means uniform convergence on compact subsets.

Definition 3. The Julia set is the complement of the Fatou set, $J_f := X \setminus F_f$.

The following questions are central in holomorphic dynamics.

(1) Describe the behavior of orbits on J_f.
(2) Describe the behavior of orbits on F_f.
(3) How do natural invariant sets such as J_f and F_f vary when the map f is perturbed.

It turns out that in order to understand the third question a good understanding of the previous two questions is needed. This is for obvious reasons, if you do not understand the sets J and F then it is unlikely that you can determine how these sets will vary as the parameters vary. It turns out that understanding question (3) can also help to understand the answers to questions (1) and (2), although

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this is more subtle. A classical example can be found in the works of Shishikura and Buff-Cheritat, who proved that there exist polynomials whose Julia sets have Hausdorff measure 2, resp. positive Lebesque measure. A more recent example is the construction of wandering Fatou components for polynomials in two complex variables.

The first two questions are closely related, and one rarely studies the Fatou set without using properties of the Julia set. In this course we will mostly focus on the second question.

We will mostly consider rational maps. Of course there are many other interesting dynamical systems, also in the holomorphic category. In fact transcendental dynamics has received significant attention in the literature. However, it is important to note that it is almost impossible to prove anything of substance about very large classes of maps, such as *all* entire maps, or *all* holomorphic automorphisms of $\mathbb{C}^2$. These classes are too large to study their dynamics (unless one wants to show that specific behavior can occur for some maps, then having a large class of maps is of course an advantage).

In the one-dimensional setting, the Riemann surface $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a particularly pleasant space to study the dynamics. The dynamical behavior of holomorphic endomorphisms is much richer there than in any other compact Riemann surface.

Homework 1. Prove that all holomorphic endomorphisms of $\hat{\mathbb{C}}$ are rational.

Homework 2. Describe the behavior of affine maps $z \mapsto az + b$. What are the Fatou and Julia sets?

The family of quadratic polynomials is one of the most studied dynamical systems. Note that we can *conjugate* any quadratic polynomial to a polynomial of the form $f(z) = z^2 + c$. Another popular form is $z \mapsto kz(1-z)$, which is called a logistic map. Of course one can easily change coordinates from one form to another. The advantage of the form $f(z) = z^2 + c$ is that it stresses the importance of the *critical point* $z = 0$, the point where the derivative vanishes, and the critical value c. We will see many examples this week that highlight the important role of the critical point in complex dynamical systems.

Homework 3. Prove that a rational function of degree d has $2d-2$ critical points, counting multiplicity.

For a polynomial of degree d the point ∞ always is a critical point of order $d-1$, meaning that the derivative vanishes at ∞ with order $d-1$. Hence there are exactly $d-1$ critical points in the complex plane, counting with multiplicity as always.

Let us look at a particular example of a quadratic polynomial. We will not choose the parameter $c = 0$, because besides being rather boring, the polynomial $f(z) = z^2$ is also quite different from any other polynomial $z^2 + c$. It will be important for us to know what happens to the orbit of the critical point 0. Let us therefore take an easy example where 0 is a periodic point. The period 1 case gives the map we do not want, so let us take a higher order, for example order 3. Now we need to solve the equation
$$f_c^3(0) = 0,$$
which is a (real) degree 4 polynomial in c, hence has 4 solutions. There are two real solutions, one of which is 0. There is also a pair of complex solutions. Let us pick one of these complex solutions, simply because the picture is nicer (at least I think so). See Figure 1.

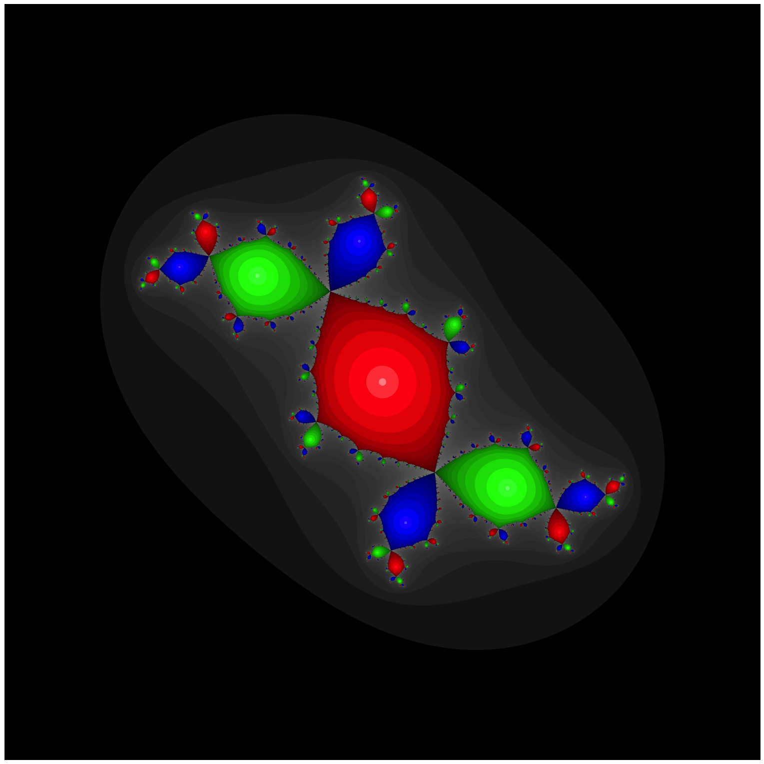

FIGURE 1. $z \mapsto z^2 + c$

What do we see in this picture? The region depicted in black (and dark grey) can be regarded as the *basin of attraction* of the fixed point at infinity,

$$I_\infty = \{z \in \mathbb{C} \mid f^n(z) \to \infty\}.$$

Note that for any polynomial there exists a constant $R > 0$ such that if $|z| \geq R$ then $|f(z)| \geq 2|z|$. It follows that if an orbit at any point leaves the disk of radius R, then it must converge to infinity. The constant R is often called the escape radius, although of course it is not unique. Drawing the black (grey) region by computer is easy. Choose coordinates for each pixel and iterate until the orbits leaves the disk $\Delta(R)$. The grey scale shows how many steps it takes to leave this disk. If the orbit has not left the disk after a large number of iterates then we do not color it grey. For a sufficiently large number of iterates this gives a very good approximation of I_∞.

The complement of I_∞ is denoted by K,

$$K := \{z \in \mathbb{C} \mid \{f^n(z)\} \text{ is bounded}\}.$$

Since I_∞ is open, the set K is closed. Both sets are completely invariant under f.

Homework 4. Prove that $J = \partial K$, and $F = K^\circ \cup \left(\hat{\mathbb{C}} \setminus K\right)$.

In the picture it appears that K is depicted in green, red and blue. This is almost true, but not quite. Note that $0 \mapsto c \mapsto f(c) \mapsto 0$. The derivative of f^3 at each of these three points is 0.

Lemma 4. *Let $f(p) = p$ and suppose that $|f'(p)| < 1$. Then there is a neighborhood $U(p)$ such that the orbit of any point in U converges to p. Conversely, suppose that $U \subset \mathbb{C}$ is an open, connected and bounded, and that $f(U)$ is a relatively compact subset of U. Then there is a point $p \in U$ with $|f'(p)| < 1$, and the orbit of any point in U converges to p.*

Homework 5. Prove the above lemma.

The points $0, c$ and $f(c)$ are attracting fixed points for f^3, hence $\{0, c, f(c)\}$ is an attracting periodic cycle for f. The attracting basin for f^3 of the point 0 is depicted in red, for c in blue, and for $f(c)$ in green. Together they form the basin A of the attracting periodic cycle of f. Note that A is again completely invariant, i.e. $f^{-1}(A) = A$.

Let us write A_0 for the connected component of A that contains 0, and write $A_1 = f(A_0)$ and $A_2 = f(A_1)$.

Homework 6. Prove that the components A_0, A_1 and A_2 are distinct, and thus disjoint.

The maps $f : A_1 \mapsto A_2$ and $f : A_2 \mapsto A_0$ are locally injective (since the critical point is not contained in A_0). By the Maximum Principle these components are simply connected. It follows that the restriction of f to either A_1 or A_2 is injective. The map $f : A_0 \to A_1$ is not injective but $2 : 1$, except that c has only 1 inverse image.

Since the polynomial f has degree 2, it follows from the above discussion that the components A_2 and A_0 each must have another pre-image. Each of these new components must also have two pre-images, and so on. It follows that the set A consists of countably many distinct connected components. The combinatorial data of these connected components gives us a complete description of all possible orbits of f.

Claim 1. $K^\circ = A$.

Claim 2. $\partial A = J = \partial$"red set" $= \partial$"blue set" $= \partial$"green set".

Claim 3. Define the sets A_{-n} for $n \in \mathbb{N}$ recursively so that $f(A_{-n}) = A_{-n+1}$, and $A_{-3} \neq A_0$. Then $\mathrm{diam}(A_{-n}) \to 0$.

Before we prove these three claims, it is worth noting that while we are looking at a very specific polynomial, these properties hold in much greater generality. Indeed, they follow from some very general results.

It is immediately clear that $\mathrm{Area}(A_{-n}) \to 0$, since the sets A_{-n} are pairwise disjoint and are all contained in $\Delta(R)$. Claim 3 is of course strictly stronger. How do we prove this claim? Note again that each component A_{-n} is simply connected and does not contain critical points. Therefore the map $f : A_{-n} \to A_{-n+1}$ is injective, and thus $f^n : A_{-n} \to A_0$ is also univalent. Therefore we can consider the inverse branches $f^{-n} : A_0 \to A_{-n}$, which are of course also univalent, and bounded since the images are contained in $\Delta(R)$. Hence these inverse branches form a normal family, and any sequence of these inverse branches must have a convergent subsequence, uniformly on compact subsets of A_0. Since all the images A_{-n} are pairwise disjoint, it follows that the image of any limit map $h = \lim f^{-n_j}$ must be a single point. After all, h must be holomorphic, and $h(A_0)$ cannot have any interior points. If it did, then some (open) set A_{-n} would intersect an interior point of $h(A_0)$, in which case that point could never be approximated by points in other sets A_{-m}.

Now we wish to conclude from the fact that any limit set $h(A_0)$ consists of a single point, that the diameter of the sets A_{-n} must converge to zero. We could conclude this immediately if the convergence was uniform. However, the convergence is only uniform, it is only uniform on compact subsets. This is not sufficient to conclude that the diameters of A_{-n} converge to zero. It is possible that the sets A_{-n} become very thin but long. How can we prove that this does not happen?

Definition 5. We say that a polynomial f is *hyperbolic* if some iterate f^N acts expandingly on the Julia set J.

Expanding means that $|f'(z)| > 1$ for any $z \in J$. By compactness of J it follows that there exists a $\mu > 1$ such that $|f'(z)| > \mu$ for any $z \in J$. An equivalent definition of hyperbolicity is that there exists a metric on J with respect to which f is uniformly expanding.

A clear example of a hyperbolic polynomial is the map $z \mapsto z^2$. The Julia set of this map is the unit circle, where the derivative has constant norm 1. It turns out that the polynomial f_c that we have been considering is also hyperbolic. Therefore it is also uniformly expanding in a given neighborhood of the Julia set. If the sets A_n get sufficiently thin then they must be contained in this neighborhood of J. But then the branches of f^{-1} that we are considering are uniform contractions on the closure of the sets A_{-n}, from which it follows that the diameters of A_{-n} converge to zero.

Looking back we notice that we actually did not need to go through the normality argument. The fact that the area convergence to zero is enough in combination with the univalence and the hyperbolicity. These two ingredients, the normality argument and the hyperbolicity, will however play a crucial role in what will come later in the course.

1.2. **A short note on Equilibrium measures.** Take any point $z \in \hat{\mathbb{C}} \setminus \{\infty\}$, and define the probability measures

$$\mu_n = \frac{1}{2^n} \sum_{f^n(w)=z} \delta_w.$$

As usual we need to sum over all w counting multiplicities.

One can show that the measures μ_n converge weakly to a probabily measure μ which is independent of the chosen point z. The measure μ is called the equilibrium measure of f. It if invariant under f and $\mathrm{supp}(\mu) = J$. Many other natural constructions lead to the same measure μ, and this measure plays an important role in our understanding of one-dimensional complex dynamical systems. The above construction works for any rational function, except for the exceptional polynomials conjugate to $z \mapsto z^d$, in which case one should not start with the two completely invariant critical points.

The construction of the equilibrium measure is due to Brolin (for polynomials, 1965), Lyubich (1982, for rational functions) and Mañé. It has been successfully generalized to higher dimensional maps, starting with the works of Bedford-Smillie (for polynomial automorphisms), Sibony-Fornæss (for holomorphic endomorphisms of projective space), and later for many more maps, for an important part due to Sibony.

1.3. **Back to the Fatou set.** Our earlier claim 2, stating that the boundary of each of the red, green and blue sets equals J, follows from the above comment that $\mathrm{supp}(\mu) = J$, but there is a much easier way to prove the claim. Let $z \in J$, and let U be a neighborhood of z. Then U must contain both red, blue and green points, otherwise the family of iterates defined on U avoids at least 2 points of $\mathbb{C}$ and by Montel's theorem is normal. But this would contradict the definition of J.

Our last claim, which was in fact the first claim, states that $A = K^\circ$, which is equivalent to saying that the Fatou set has no other connected components. Connected components of the Fatou set are called Fatou components, and for rational functions they are very well understood. The proof of the last claim will follow from the following two theorems.

Theorem 6 (Fatou, Siegel, Herman, Sullivan)**.** *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be rational of degree at least* 2*, and let U be a Fatou component. Then U is either periodic or pre-periodic. If U is invariant then U must either be*

- Attracting basin *the immediate basin of an attracting fixed point p, i.e. $f(p) = p$ and $|f'(p)| < 1$.*
- Parabolic basin *an immediate basin of a parabolic fixed point p, i.e. $f(p) = p$ and $f'(p) = 1$.*
- Siegel disk*: equivalent to a disk on which f acts as an irrational rotation.*
- Herman ring*: equivalent to an annulus on which f acts as an irrational rotation.*

The classification is due to Fatou, but he was not able to decide whether the third and fourth cases could exist. The existence of Siegel disks was shown by Siegel in the 1940's, the existence of Herman rings was shown by Herman in 1979. In both cases there exists a conformal map $\varphi : U \mapsto \Sigma$ such that $\phi(f(z)) = \lambda \cdot \phi(z)$, where $\lambda = e^{2\pi i \theta}$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Here Σ is either the unit disk (in the Siegel case) or an annulus (in the Herman case). Note that Herman rings cannot exist for polynomials. It was shown by Sullivan in 1985 that every Fatou component is either periodic or pre-periodic. A Fatou component that is not periodic or pre-periodic is called *wandering*. For this reason one often refers to Sulivan's No Wandering Domains Theorem.

Theorem 7 (Fatou components and critical points)**.** *The immediate basin of an attracting fixed point or attracting parabolic point must contain a critical point. The boundary of a Siegel disk must be contained in the closure of a critical orbit.*

Since for our specific polynomial the critical orbit is a finite (and thus compact) subset of $A_0 \cup A_1 \cup A_2$ it follows that there can be no invariant Siegel disks, parabolic basins or other attracting basins. This not only holds for f but also for f^n for any n. Hence there can be no other periodic Fatou components, and it follows from Sullivan's No Wandering Domains Theorem that there exist no other Fatou components, hence $A = K^\circ$.

2. Lecture 2, classification of Fatou components

In this lecture we will prove the classification of one-dimensional Fatou components discussed in the previous lecture. We will cheat a little bit and work with polynomials instead of rational functions. In the rational case one would need to work with hyperbolic Riemann surfaces, where we will have an easier time working with bounded simply connected domains in $\mathbb{C}$. A more complete discussion can be found in Milnor's book *Dynamics in One Complex Variable*.

Recall that there exists a conformal metric on the unit disk that is invariant under all conformal automorphisms of $\mathbb{D}$. This metric is unique up to a multiplicative constant and is called the Poincaré metric. The precise formula of the Poincaré metric will not be important for us here, nor will the choice of the multiplicative constant. By the Riemann mapping theorem we can push forward the Poincaré metric to any bounded simply connected domain $U \subset \mathbb{C}$. We will also refer to this metric on U as the Poincaré metric.

Homework 7. Prove that the Poincaré metric on U is independent of the choice of the Riemann mapping.

Now let U and V both be bounded simply connected domains in $\mathbb{C}$, and let $f : U \to V$ be holomorphic. Then f is non-expanding with respect to the Poincaré metrics of U and V. In fact, either f is biholomorphic, and thus an isometry, or f strictly reduces Poincaré distances, uniformly on compact subsets of U.

Theorem 8. *Let U be a bounded simply connected domain and let $f : U \to U$ be holomorphic. Then one of the following holds*

- *There is an attracting fixed point and all orbits converge to this point, uniformly on compact subsets of U.*
- *The map f is conjugate to an irrational rotation.*
- *There is an $k \in \mathbb{N}$ such that $f^k = \mathrm{Id}$.*
- *The map is escaping, i.e. for any compact $K \subset U$ there exists an N_K such that $f^n(K) \cap K = \emptyset$ for $n \geq N_k$.*

This escaping case is sometimes called *compactly divergent.*

Proof. We may assume that we are dealing with the unit disk $U = \mathbb{D}$. Suppose that f is not escaping. If f preserves the Poincaré metric then f is an automorphism of $\mathbb{D}$, and thus of the form

$$z \mapsto \theta \cdot \frac{z-a}{1-\overline{a}z}.$$

The only maps of this form that are not escaping are conjugate to rotations, either rational or irrational, which gives either the second or the third case.

If f strictly decreases Poincaré distances, uniformly on compact subsets of U, then it follows that there is a $r < 1$ and an integer k such that $f^k(\Delta(r))$ is relatively compact in $\Delta(r)$, and hence f^k must have an attracting fixed point p in $\Delta(r)$. Since f^k strictly decreases Poincaré distances it can only have one fixed point, and hence p must also be an attracting fixed point of f. This completes the proof. $\square$

Homework 8. Prove that all holomorphic automorphisms of $\mathbb{D}$ are indeed of the form

$$z \mapsto \theta \cdot \frac{z-a}{1-\overline{a}z}.$$

One has to be a little careful when assuming that $U = \mathbb{D}$. In the above proof all goes well, but in the next result it can lead to a proof that is correct for the unit disk but does not hold in general.

Lemma 9. *Let $f : U \to U$ be escaping and suppose that f extend continuously to the boundary ∂U. Also assume that $f|_{\partial U}$ has only finitely many fixed points. Then f^n converges to a single boundary point $p \in U$,* In the case $U = \mathbb{D}$ the assumption that f extends continuously to the boundary is redundant, and without this assumption the result is known as the Denjoy-Wolff theorem. The Denjoy-Wolff theorem does hold in greater generality, even outside the holomorphic setting and in also in several complex variables, but one needs to assume some kind of convexity property. It does *not* hold for all simply connected domains! So we will really need that f extends continuously to the boundary, and we may not assume that $U = \mathbb{D}$.

Proof. Take a point $z \in U$, and suppose that $f^{n_j}(z) \to p \in \partial U$ for some sequence (n_j). Since the Poincaré metric blows up near the boundary of U, it follows that $f^{n_j+1}(z)$ also converges to p, which implies that $f(p) = p$. By the assumption that f has only finitely many fixed points in ∂U there can be at most finitely many such limit points p. Now let us assume for the purpose of a contradiction that there exist at least two limit points p and q.

Let $\gamma : [0, 1] \to U$ be smooth with $\gamma(0) = z$ and $\gamma(1) = f(z)$. Then we can extend γ to the positive real line by recursively defining $\gamma(x + 1) = f(\gamma(x))$. The piecewise smooth curve will converge to ∂U, and will accumulate both at p and q. Now take a small disk $D_\epsilon(p)$, so that q lies outside of the disk. Then the curve must pass through the boundary of this disk infinitely many times, or to be more precise, $\gamma(x) \in \partial D_\epsilon(p)$ for arbitrarily large values of x. Again, since the Poincaré metric blows up near ∂U, it follows that some subsequence $f^{m_l}(z)$ must converge to a boundary point of this disk. By choosing the radius of the disk carefully we can make sure no fixed points on $\partial U \cap \partial D_\epsilon(p)$. B passing to a convergent subsequence of f^{m_l} we obtain a counterexample. $\square$

Lemma 10. *Let f be a polynomial and U an invariant Fatou component, and assume that $f^n|_U$ converges to $p \in \partial U$. Then $f'(p) = 1$.*

For the proof of this result we also do not really need that the map is a global polynomial, but we do need that f is holomorphic in a neighborhood of p.

Proof. Note that the fixed point p cannot be attracting, since attracting fixed points lie in the interior of their basin. Hence $|f'| \geq 1$, and in particular $f'(p) \neq 0$.

By conjugating with a translation we may assume that $p = 0$. Let V_0 be a relatively compact subset of U that is both open and simply connected. The iterates f^n converge to p uniformly on V_0. By choosing V_0 sufficiently large we may assume that $f(V_0) \cap V_1 \neq \emptyset$, where $V_n = f^n(V_0)$. Finally, we may assume that all the sets V_n are contained in a small disk centered at the fixed point 0 on which f acts injectively.

Now let us suppose for the purpose of a contradiction that $f'(0) \neq 1$. Let $w_0 \in V_0$ such that $w_1 = f(w_0)$ lies in V_0 as well. $\varphi : \mathbb{D} \to V_0$ be such that $\varphi(0) = w_0$. We define the maps $g_n : \mathbb{D} \to V_n$ by

$$g_n(\zeta) = \frac{f^n \circ \varphi(\zeta)}{f^n \circ \varphi(0)} = \frac{f^n \circ \varphi(\zeta)}{w_n}.$$

By our assumptions the maps g_n are all conformal. Also, $g_n(0) = 1$ for all $n \in \mathbb{N}$, while $0 \notin g_n(\mathbb{D})$ for all $n \in \mathbb{N}$. Hence by the Koebe Distortion Theorem the family $\{g_n\}$ is normal. Now notice that

$$g_n(\varphi^{-1}(w_1)) = \frac{w_{n+1}}{w_n} \to f'(0),$$

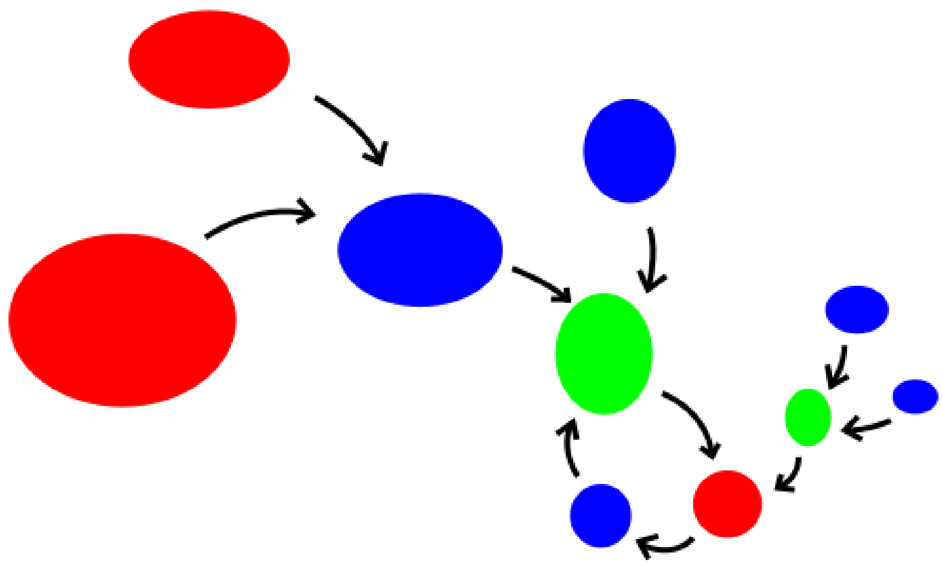

FIGURE 2. All Fatou components are periodic or pre-periodic

and by assumption $f'(0) \neq 1$. Recall that the limit of a convergent sequence of conformal functions must either be degenerate (i.e. mapping to a point) or be conformal as well. But it cannot be degenerate, since $g_n(0) = 1$ while $g_n(\varphi^{-1}(w_1)) \to f'(0)$. Therefore all limits of the sequence (g_n) are conformal, and there must be a bound from below on the derivatives $|g'(0)|$, say

$$|g_n'(0)| > \epsilon.$$

Hence it follows by the Koebe-$\frac{1}{4}$ Theorem that $g_n(\mathbb{D})$ contains $D_{\frac{\epsilon}{4}}(1)$, which implies that V_n contains a disk centered at w_n of radius $\frac{\epsilon}{4} \cdot |w_n|$. In other words, the inner radius of the sets V_n are at least comparable to the distance to the origin.

Note that $|f'(0)|$ has to be equal to 1. We already remarked that it cannot be smaller, as otherwise 0 would be attracting. But the norm also cannot be larger than 1, otherwise orbits could not converge to the origin. By our assumption that $f'(0) \neq 1$ it follows that multiplication by $f'(0)$ gives a rotation. As the sets V_n approach the origin, the maps f starts to act more and more as the linear rotation. It follows that the points w_n can only approach the origin at a slower and slower rate, while they keep rotating around 0 at a fixed rate. Combining these facts with our estimate on the inner radius of the sets V_n gives that at some point one can find a simple closed curve that winds around the origin and is contained in a finite number of sets V_n. Hence f^n converges to 0 uniformly on this closed curve, and by the Maximum Principle it follows that f^n converges to 0 uniformly in a neighborhood of the origin. But this contradicts the fact that the fixed point lies on the boundary of the Fatou component. $\square$

Now we have completed the classification of invariant Fatou components. We will later get back to Sullivan's Wandering Domains Theorem for specific classes of polynomials, both in one and in several variables.

Lemma 11. *If U is the immediate basin of an attracting fixed point, then U must contain a critical point.*

Proof. If not then $f : U \mapsto U$ must be an automorphism and thus preserve the Poincaré metric. But isometries cannot have attracting fixed points. □

Here is another argument with which one can show that U must contain a critical point. Let us first assume that the attracting fixed point $p \in U$ is not critical itself, so that $\lambda := f'(p) \neq 0$. Then define the maps $\varphi_n : U \to \mathbb{C}$ by

$$\varphi_n(z) = \lambda^{-n} \cdot f^n.$$

Let $K \subset U$ be compact.

Homework 9. Prove that there exist $0 < r < 1$ and $C > 0$ so that

$$|\varphi_{n+1}(z) - \varphi_n(z)| < Cr^n.$$

It follows that the sequence (φ_n) converges to a limit map $\Phi : U \to \mathbb{C}$, uniformly on compact subsets of U. One has $\Phi'(0) = 1$ thus Φ is locally invertible in the origin. The fact that the map $f : U \to U$ is proper implies that Φ is also surjective. If f has no critical points then it is a conformal map from $U \to \mathbb{C}$, which gives a contradiction since U is equivalent to the unit disk.

In the parabolic case one can give a quite similar argument to show that the immediate basin must contain a critical point. If not, then the parabolic basin must be equivalent to $\mathbb{C}$. We will come back later to the role that critical points play near the boundary of Siegel disks.

3. Lecture 3, classification of invariant Fatou components in higher dimensions

So far we have mostly studied the dynamics of polynomials in one complex variable. We will now switch to higher dimensions. However, it is not a good idea to attempt to study all higher dimensional maps at the same time, the differences between different polynomial maps are too great. For example, in one variable the algebraic degree of a polynomial equals its topological degree. In higher dimension this is not true, consider for example the following three maps.

(1) $(z, w) \mapsto (z, w + p(z)$ (a *shear*)
(2) $(z, w) \mapsto (z, w^2)$
(3) $(z, w) \mapsto (z^2, w^2)$

While these three polynomials all have algebraic degree 2, but the topological degrees are respectively 1, 2 and 4.

Even more curious are the degrees of iterates. If f is a rational function of degree d, then the degree of f^n is d^n. Not true in higher dimensions. For instance, the iterates of the polynomial $(z, w) \mapsto (zw, z)$ have degrees equal to the Fibonacci numbers!

While it is not reasonable to try to describe the dynamical behavior of all of the above maps at the same time, there are several interesting classes of higher dimensional rational maps. The class of maps that has received the most attention is the family of polynomial automorphisms of $\mathbb{C}^2$. This is the class that we will focus on from now. The class of holomorphic endomorphisms of $\mathbb{P}^2$ is also interesting, and it can be considered the most natural generalization of one dimensional rational functions. A third class that I would like to mention is the family of polynomial skew products, i.e. maps of the form $(z,w) \mapsto (f_w(z), g(w))$. These skew products map vertical planes to vertical planes by one-dimensional polynomials. This means that one-dimensional techniques can be used to study their properties. Polynomial skew products have often been used to construct examples of maps with dynamical behavior that was at the time unknown to exist.

3.1. **Hénon maps.** In the 1970's the theoretical astronomer Michel Hénon suggested a family of two-dimensional polynomial maps as a good model for the more complicated Lorenz model. These maps were particular invertible polynomial automorphisms of $\mathbb{R}^2$, but they were also be studied in the complex category. In 1989, Friedland and Milnor proved that the only polynomial automorphisms of $\mathbb{C}^2$ with interesting dynamical behavior are conjugate to finite compositions of generalized Hénon maps, i.e. maps of the form

$$H(z,w) = (p(z) + \delta w, z),$$

where p is a polynomial of degree at least 2, and $\delta \neq 0$. While compositions of such maps are far more general than the maps suggested by Hénon, we will refer to them as Hénon maps.

The proof of Friedland and Milnor relies on a classical result of Jung from the 1940's, which states that the polynomial automorphism group of $\mathbb{C}^2$ is generated by affine maps and shears. Compositions of affine maps and shears are called tame. So Jung's Theorem says that any polynomial of $\mathbb{C}^2$ is tame. It was shown much more recently, in 2003 by Shestakov and Umirbaev, that there exist polynomial automorphisms of $\mathbb{C}^3$ that are not tame, although their example was already suggested by Nakana in the 1970's.

Note the significant differences between a shear

$$F(z,w) = (z, w + p(z))$$

and a seemingly similar Hénon map

$$H(z,w) = (w + p(z), z).$$

While the only change is the switching of the two coordinate functions, their dynamical behavior is vastly different. For example, the iterates of the shear can be written in the closed form $F^n = (z, w + np(z))$, while this is not possible for iterates of H. The algebraic degree of F^n is equal to the degree of F, while the algebraic degree of H^n is the n-th power of the degree of H. The map F can be written as the time-1 map of a flow, while the map H cannot even be written as a second iterate, a result of Fornæss and Buzzard from 2003.

It turns out that one can still say a lot about the dynamics of Hénon maps, see for example the long and ongoing series of articles that Bedford and Smillie have written on the topic.

3.2. **The filtration.** For the rest of this lecture let $f = H_1 \circ \ldots \circ H_j$ be a Hénon map of algebraic degree $d \geq 2$. For $R > 0$ we define the following sets:

$$\begin{aligned} V^+ &= \{(z,w) \in \mathbb{C}^2 \mid |z| \geq \max(|w|, R)\} \\ V^- &= \{(z,w) \in \mathbb{C}^2 \mid |w| \geq \max(|z|, R)\} \\ \Delta^2(R) &= \{(z,w) \in \mathbb{C}^2 \mid |z|, |w| \leq R\} \end{aligned}$$

Homework 10. Show that $R > 0$ can be chosen sufficiently large so that

(1) $f(V^+) \subset V^+$

(2) $f^{-1}(V^-) \subset V^-$
(3) If $z \in V^\pm$ then
$$\|f^{\pm 1}(z)\| \geq 2\|z\|.$$

It follows from these properties that any forward orbit that reaches V^+ must diverge to infinity. In fact, the orbit has to converge to the point $[1:0:0]$ on the line at infinity in $\mathbb{P}^2$. On the other hand, a forward orbit that does not diverge to infinity must at some point reach $\Delta^2(R)$ and never leave it. The radius R plays the role of the escape radius as we used it in for polynomials in one complex variable. Here, just like there, we have the property that most "interesting dynamics" takes place in a compact subset, which is very useful.

Continuing the analogy with one-dimensional dynamics we define the basin of infinity as
$$\begin{aligned} I_\infty &= \{z \in \mathbb{C}^2 \mid f^n(z) \to \infty\} \\ &= \bigcup f^{-n}(V^+). \end{aligned}$$
It follows immediately that I_∞ is open, connected and completely invariant. Similarly we define the set of bounded orbits both for forward and backwards time:
$$K^\pm = \{z \in \mathbb{C}^2 \mid \{f^n(z)\} \text{ bounded}\}.$$
The forward and backwards Julia sets are then defined by $J^\pm = \partial K^\pm$, and the Fatou sets by $F^\pm = \mathbb{C}^2 \setminus J^\pm$. Just as in one variable we can equivalently define both Fatou sets as the set of local normality for the family of forwards or backwards iterates, using the restrictions to $\mathbb{C}^2$ of a metric defining the topology on $\mathbb{P}^2$. In this course we are again interested in describing the possible Fatou components, i.e. connected components of F^+.

3.3. **A short comment on the equilibrium measure.** The construction of the equilibrium measure used in one variable cannot literally be used in higher dimensions, as a point has only a single inverse image. While there are still ways to define the measure using equidistribution, we will use a different construction here, due to Bedford-Smillie. Define the forward and backwards Green's functions by
$$G^\pm := \lim_{n\to\infty} \frac{1}{d^n} \cdot \log^+ \|f^{\pm n}(z)\|.$$
These two pluri-subharmonic functions play a very important role in our understanding of Hénon maps and will come back later. Let us point out right away that they satisfy the functional equations
$$G^\pm(f(z)) = d^{\pm 1} G^\pm(z).$$

Let us now define the positive, closed $(1,1)$ currents by
$$T^\pm = 2i\partial\overline{\partial}G^\pm.$$
The support of the currents $T^\pm$ is equal to the Julia sets $J^\pm$. Now define
$$\mu = T^+ \wedge T^-.$$
It was shown by Bedford and Smillie that μ is a probability measure whose support is contained in $J = J^+ \cap J^-$. Whether its support is always equal to J is an important open question, currently only settled for hyperbolic Hénon maps.

3.4. **Non-escaping Fatou components.** In the description of Fatou components we are many years behind the one dimensional situation. The existence of wandering Fatou components is open, but in the fifth lecture we will consider some families of Hénon maps for which the non-existence of wandering Fatou components is known.

Pre-periodic Fatou components do not exist since we are working with invertible maps. In what follows we will attempt to describe the state of knowledge regarding invariant Fatou components.

As for polynomials in one variable there always exists a single basin of infinity I_∞. We will focus on the rest, which only contain orbits with bounded forward orbits. Recall that any such orbit must eventually reach $\Delta^2(R)$.

It is useful to immediately make a distinction between escaping and the non-escaping Fatou components. In the former case all orbits converge to the boundary, while in the latter case, which is usually called the recurrent case, every orbit is contained in a compact subset. The reason for the name recurrent is that for these invariant Fatou components there exists a recurrent orbit.

Theorem 12 (Bedford-Smillie 1991)**.** *Let U be a non-escaping invariant Fatou component with bounded forward orbits. Then either*

(1) (Attracting) *all orbits converge to an attracting fixed point $p \in U$, and $U \cong \mathbb{C}^2$.*
(2) (Attracting-rotating) *all orbits converge to a* 1*-dimensional submanifold Σ which is closed in U. The Riemann surface Σ is either an embedded disk or an annulus, and f acts on Σ as an irrational rotation.*
(3) (Siegel domain) *there exists a sequence f^{n_j} converging to the identity.*

There are still a number of open questions in the non-escaping case. For example, it is not known whether the Riemann surface Σ can actually be an annulus. The only known examples of Siegel domains are those arising from linearizable fixed points. In particular it is not known if a Siegel domain can have non-trivial topology. Can a Siegel domain be biholomorphic to an annulus cross a disk?

There is a similar result for non-escaping Fatou components of holomorphic endomorphisms by Fornæss and Sibony (1994). In this case the attracting-rotating annulus can exist, but Ueda ruled out an attracting-rotating punctured disk in 2008.

Time does not permit us to discuss the proof of the classification of non-escaping Fatou components here. Instead we will focus on the escaping case.

3.5. **The escaping case.** Note that the Jacobian determinant of a polynomial automorphism is a non-constant polynomial on $\mathbb{C}^2$, and is therefore constant. Hence a Hénon map is either uniformly volume expanding, volume contracting ("dissipative") or volume preserving. In the volume expanding case it follows from the existence of the filtration that a volume expanding Hénon map does not have any Fatou components besides the basin of infinity. In the volume preserving case all other Fatou components are Siegel domains. In particular, and this follows immediately from the fact that $\Delta^2(R)$ has finite volume, there are no wandering Fato components. The dissipative case is most interesting and the least understood. The non-escaping invariant Fatou components are attracting or attracting-rotating. What about the invariant components that are escaping?

Let f be an Hénon map of degree d and with Jacobian determinant δ. We will say that f is *substantially dissipative* if

$$|\delta| < \frac{1}{d^2}.$$

Theorem 13. *Let f be a substantially dissipative Hénon map and let U be an escaping invariant Fatou component. Then the maps f^n converge uniformly on compact subsets of U to a fixed point $p \in \partial U$. The eigenvalues λ_1, λ_2 of $Df(p)$ satisfy $\lambda_1 = 1$ and $|\lambda_2| < 1$. In particular $U \cong C^2$.*

The hypothesis that f is substantially dissipative is essential in the proof, and it is not known whether the above result holds when $\frac{1}{d^2} \leq |Jac(f)| < 1$. It is however clear that in order to possibly prove the result in this *near conservative* case, a new idea is needed.

3.6. **Stable manifolds.** Attracting sets occur in many different variations. The most basic example is the attracting basin of a single attracting fixed point. If a fixed point $p = f(p)$ is not attracting but *hyperbolic*, meaning that the eigenvalues λ_1, λ_2 of $Df(p)$ satisfy $|\lambda_1| < 1 < |\lambda_2|$, then one can still define

$$\Sigma^s_f(p) = \{z \in \mathbb{C}^2 \mid f^n(z) \to p\},$$

but the set $\Sigma^s_f(p)$ does not contain an open neighborhood of p. In fact, $\Sigma^s_f(p)$ is a one-dimensional complex manifold, or to be more precise, an injectively immersed Riemann surface, equivalent to the complex plane. Note that if f is linear then $\Sigma^s_f(p)$ is a straight complex line, determined by eigen vector v_1 corresponding to the attracting eigenvalue λ_1. If f is a Hénon map then $\Sigma^s_f(p)$ is never a straight line, and is generally not a submanifold. However, $\Sigma^s_f(p)$ is locally a graph over the plane spanned by the vector v_1. Here it is important that by "locally" we mean *locally in the stable manifold*, not locally in the ambient space $\mathbb{C}^2$.

In general, if an automorpism $F : X^m \to X^m$ has a fixed point p and $Df(p)$ has k attracting and $m-k$ repelling generalized eigen values, then the stable and unstable manifolds are locall defined, and are complex manifolds of dimension k and $m-k$. The stable and unstable manifolds are equivalent to $\mathbb{C}^k$ resp. $\mathbb{C}^{m-k}$.

If F has some *neutral* eigenvalues then one can still define the stable manifold by choosing an $r < 1$ which is strictly larger than all attracting eigen values, and writing

$$\Sigma^s_f(p) = \{z \in C \mid \exists C > 0 \forall n \in \mathbb{N} : d(f^n(z), p) < Cr^n\},$$

which is still a complex manifold of dimension equal to the number of attracting eigen values, and is sometimes called the strong stable manifold. Note that the constant C is allowed to depend on the point z.

Stable and unstable manifolds can also be defined when the point p is not fixed but lies in some compact invariant subset $K \subset X$ and there is an *invariant splitting* of the tangent bundle into attracting and repelling directions. One then defines

$$\Sigma^s_f(p) := \{z \in X \mid \exists C > 0 \forall n \in \mathbb{N} : d(f^n(z), f^n(p)) < Cr^n\}.$$

These stable and unstable manifolds are still complex manifolds, however, if they have dimension $k \geq 2$ then it is in general not known whether they are equivalent to $\mathbb{C}^k$. This open question is known as the Bedford Conjecture.

Let us go back to Hénon maps, in which case the stable and unstable manifolds are always 1-dimensional and equivalent to the complex plane. Let us discuss how the biholomorphism from $\mathbb{C}$ to a stable manifold can be defined. For convenience we will work with the stable manifold of a fixed point, but the construction can be used when p is not periodic as well.

As we said before, the complex manifold is locally a graph over the a complex plane spanned by the attracting eigen direction, corresponding to the attracting eigen value λ. Denote the local projection from this plane to the stable manifold by π_s, and define $\varphi_n : \mathbb{C}_\zeta \to \Sigma^s_f(p)$ by

$$\varphi_n(\zeta) = f^{-n} \circ \pi_s(\lambda^n \zeta).$$

Then the maps φ_n converge uniformly on compact subsets of $\mathbb{C}$ to a biholomorphic map $\varphi : \mathbb{C} \to \Sigma^s_f(p)$. The map φ is called the linearization map, and satisfies the functional equation

$$\varphi(\lambda \cdot \zeta) = f \circ \varphi(\zeta).$$

The linearization map is unique up to a multiplicative constant.

3.7. **Substantially dissipative Hénon maps.** Recall that the backwards Green function G^- is a plurisubharmonic function on $\mathbb{C}^2$ that satisfies

$$G^-(f(z)) = \frac{1}{d} G^-(z).$$

Also recall that a Hénon map is called substantially dissipative if

$$|\delta| = |\mathrm{Jac}(f)| < \frac{1}{d^2}.$$

Homework 11. Let p be a hyperbolic fixed point of a substantially dissipative Hénon map, and let φ be its linearization map. Then $g := G^- \circ \varphi$ is a non-constant subharmonic function defined on all of $\mathbb{C}$ that near infinity satisfies

$$g(z) = O(|z|^r)$$

for some $r < \frac{1}{2}$.

We say that the function g is of order of growth less than $\frac{1}{2}$.

Theorem 14 (Wiman)**.** *If $g : \mathbb{C} \to \mathbb{R}$ is a non-constant subharmonic function of order of growth less than $\frac{1}{2}$. Then all connected subsets of sublevel sets $\{g < c\}$ are bounded.*

In fact Wiman's classical result fromt he 1920's says more than this, but this is the property that is used to classify escaping Fatou components.

4. Lecture 4, Proof of invariant Fatou components

4.1. **Smoothness of limit sets.** Let U be an escaping invariant Fatou component of our Hénon map f. Our first goal is to prove that all orbits must converge to a single boundary point. By normality there exist many convergent subsequences f^{n_j} on U. A limit map $h = lim f^{n_j}|_U$ maps into the boundary and therefore a priori either has rank 0 or rank 1 (in the rank 1 case the rank can drop to 0 in isolated points). Note that in the rank 0 case the image is a single point. In the rank 1 case the image is a possibly singular analytic set. The first step is to show that the image cannot be singular.

Lemma 15. *Let $f : X^2 \to X^2$ be a holomorphic endomorphism, let U be an invariant Fatou component that is escaping, and let (f^{n_j}) be a sequence of iterates that converges uniformly on compact subsets of U to a limit map $h : U \to \partial U$. Suppose that h has generic rank 1. Then $h(U)$ is an injectively immersed Riemann surface.*

Proposition 16. *Let $h : \mathbb{D} \to \mathbb{C}^2$ be holomorphic. If $h(\mathbb{D})$ is singular then there exists an $\epsilon > 0$ such that if $h' : \mathbb{D} \to \mathbb{C}^2$ is holomorphic and satisfies $\|h - h'\|_{\mathbb{D}} < \epsilon$ then $h(\mathbb{D}) \cap h'(\mathbb{D}) \neq \emptyset$.*

Homework 12. Prove this proposition for the standard cusp
$$h(t) = (t^2, t^3).$$

Prove that lemma follows from proposition.