Solution Phase Space and Conserved Charges

A general formulation for charges associated with exact symmetries

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1. A review on covariant phase space method

2. Solution phase space

3. Conserved charges associated with exact symmetries
   - K. Hajian and M.M. Sheikh-Jabbari, *Solution phase space and conserved charges, A general formulation for charges associated with exact symmetries*, Phys.Rev. D**93** (2016) 4, 044074, [arXiv:1512.05584].

4. Application: Kerr-Newman-(A)dS charges and first law(s)
   - K. Hajian, *Conserved Charges and First Law of Thermodynamics for Kerr-de Sitter Black Holes,* [arXiv:1602.05575].

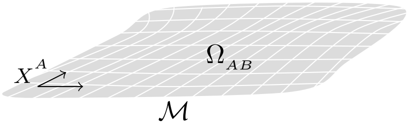
1 A review on covariant phase space method

Phase space is a manifold $\mathcal{M}$ equipped with Ω_{AB} such that:

- $\Omega_{AB} = -\Omega_{BA}$
- $\delta\Omega = 0$
- $\Omega_{AB} V^B = 0 \;\Leftrightarrow\; V^B = 0$

- $\Omega^{AB} = (\Omega^{-1})_{AB}$

Phase Space



Manifold $\mathcal{M}$ with symplectic 2-form Ω_{AB}

- C. Crnkovic and E. Witten, *Covariant Description Of Canonical Formalism In Geometrical Theories*, In Hawking, S.W. (ed.), Israel, W. (ed.): Three hundred years of gravitation,(1987), 676-684.
- Abhay Ashtekar, Luca Bombelli, and Rabinder Koul, *Phase space formulation of general relativity without a 3+1 splitting*, Lect. Notes Phys., **278**, , (1987).
- J. Lee and R. M. Wald, *Local symmetries and constraints*, J. Math. Phys. **31**, 725 (1990).

Phase space manifold $\mathcal{M}$

$\mathcal{S} = \int \mathbf{L}[\Phi] \quad \Rightarrow \quad$ phase space manifold $\mathcal{M}$ is composed of field $\Phi(x^\alpha)$.

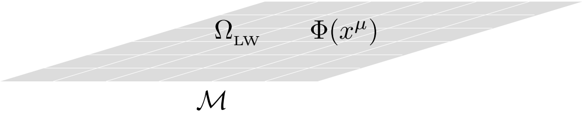
Symplectic 2-form Ω

$\delta\mathbf{L}[\Phi] = \mathbf{E}(\Phi)\delta\Phi + \mathrm{d}\boldsymbol{\Theta}(\delta\Phi, \Phi)$, then the symplectic form Ω would be:

$$\Omega(\delta_1\Phi, \delta_2\Phi, \Phi) = \int_\Sigma \boldsymbol{\omega}_{\text{LW}}(\delta_1\Phi, \delta_2\Phi, \Phi)\,,$$

in which $\boldsymbol{\omega}_{\text{LW}} = \delta_1\boldsymbol{\Theta}(\delta_2\Phi, \Phi) - \delta_2\boldsymbol{\Theta}(\delta_1\Phi, \Phi)$.

Covariant phase space



Independence of Ω from Σ (conservation)

$$\mathrm{d}\boldsymbol{\omega}(\delta_1\Phi, \delta_2\Phi, \Phi) = 0 \qquad \text{and} \qquad \boldsymbol{\omega}(\delta_1\Phi, \delta_2\Phi, \Phi)\Big|_{\partial\Sigma} = 0\,,$$

- the former is satisfied if Φ and $\delta\Phi$ satisfy e.o.m and linearized e.o.m respectively
- for the latter, one usually needs to impose some (fall-off) conditions on $\delta\Phi$.

Ambiguity

$$\begin{aligned} &\boldsymbol{\Theta}(\delta\Phi, \Phi) \to \boldsymbol{\Theta}(\delta\Phi, \Phi) + \mathrm{d}\mathbf{Y}(\delta\Phi, \Phi) \\ \Rightarrow \quad &\boldsymbol{\omega}(\delta_1\Phi, \delta_2\Phi, \Phi) \to \boldsymbol{\omega}(\delta_1\Phi, \delta_2\Phi, \Phi) + \mathrm{d}\big(\delta_2\mathbf{Y}(\delta_1\Phi, \Phi) - \delta_1\mathbf{Y}(\delta_2\Phi, \Phi)\big) \end{aligned}$$

Hamiltonian generators (conserved charges)

Hamiltonian generator variation associated with ϵ

For any diff+gauge generator $\epsilon \equiv \{\xi, \lambda\}$ in spacetime (with arbitrary $\delta\epsilon$), the Hamiltonian generator is:

$$\delta H_\epsilon \equiv \int_\Sigma \delta^{[\Phi]}\boldsymbol{\Theta}(\delta_\epsilon\Phi, \Phi) - \delta_\epsilon\boldsymbol{\Theta}(\delta\Phi, \Phi) = \int_\Sigma \mathrm{d}\boldsymbol{k}_\epsilon(\delta\Phi, \Phi) = \oint_{\partial\Sigma} \boldsymbol{k}_\epsilon(\delta\Phi, \Phi)\,.$$

Conservation conditions

$$\mathrm{d}\boldsymbol{\omega}(\delta\Phi, \delta_\epsilon\Phi, \Phi) \approx 0\,, \qquad \boldsymbol{\omega}(\delta\Phi, \delta_\epsilon\Phi, \Phi)\Big|_{\partial\Sigma} \approx 0\,.$$

Integrability condition

$$(\delta_1\delta_2 - \delta_2\delta_1)H_\epsilon \approx 0\,, \quad \forall\,(\delta_1\Phi, \delta_2\Phi, \Phi)$$

$$\Rightarrow \quad \oint_{\partial\Sigma} \Big(\xi \cdot \boldsymbol{\omega}(\delta_1\Phi, \delta_2\Phi, \Phi) + \boldsymbol{k}_{\delta_1\epsilon}(\delta_2\Phi, \Phi) - \boldsymbol{k}_{\delta_2\epsilon}(\delta_1\Phi, \Phi)\Big) = 0.$$

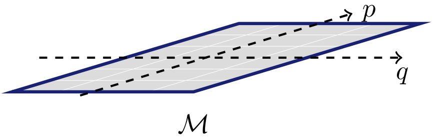
- G. Compre, P. J. Mao, A. Seraj and M. M. Sheikh-Jabbari, *Symplectic and Killing symmetries of AdS_3 gravity: holographic vs boundary gravitons*, JHEP **1601**, 080 (2016). [arXiv:1511.06079].

2 Solution phase space

Microcanonical ensemble

N free particles in a given volume and energy interval.

$$S = -k_B \ln \left(\mathrm{Vol}(\mathcal{M})^N\right)$$

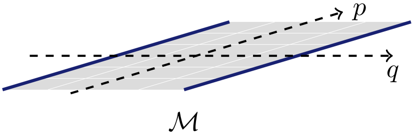


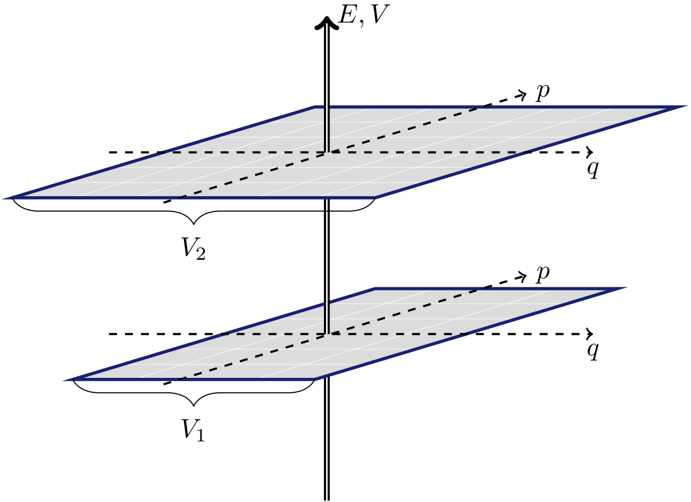
Canonical ensemble

N free particles in a given volume and temperature.

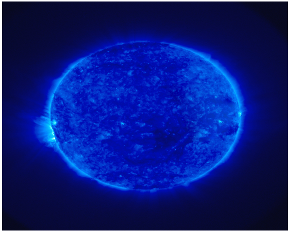
$$Z = \int_{\mathcal{M}_N} e^{-\beta E}$$

$$S = k_B (1 - \beta \frac{\partial}{\partial \beta}) \ln Z$$

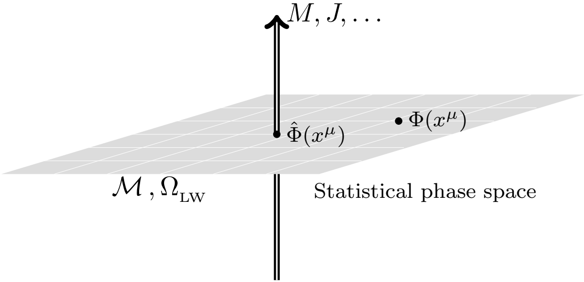




A useful diagram: phase space of a system at different thermodynamic variables



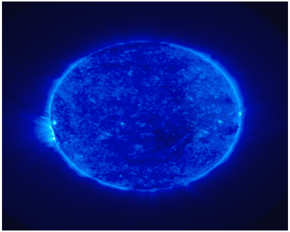
A (black Hole) solution identified by its dynamical fields $\hat{\Phi}(x^\mu)$



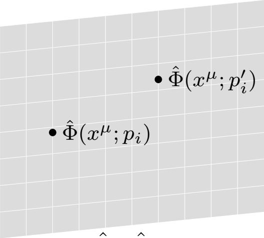
- The $\Phi(x^\mu)$ are fields generated from the known black hole $\hat{\Phi}(x^\mu)$ by the action of non-trivial diffeomorphism generators.

- G. Compère, K. Hajian, A. Seraj, and M. M. Sheikh-Jabbari, *Extremal Rotating Black Holes in the Near-Horizon Limit: Phase Space and Symmetry Algebra*, Physics Letters B, **749**, (2015). [arXiv:1503.07861].
- G. Compère, K. Hajian, A. Seraj, and M. M. Sheikh-Jabbari, *Wiggling Throat of Extremal Black Holes*, JHEP **1510** (2015) 093. [arXiv:1506.07181].
- G. Compre, P. J. Mao, A. Seraj and M. M. Sheikh-Jabbari, *Symplectic and Killing symmetries of AdS_3 gravity: holographic vs boundary gravitons*, JHEP **1601**, 080 (2016). [arXiv:1511.06079].

Covariant *solution* phase space



Solutions identified by some parameters $\hat{\Phi}(x^\mu; p_i)$



$\hat{\mathcal{M}}\,,\,\hat{\Omega}_{\text{LW}}$

- The set of solutions $\hat{\Phi}(x^\mu; p_i)$ constitute a covariant phase space, which we have called "solution phase space". The manifold $\hat{\mathcal{M}}$ is built of $\hat{\Phi}(x^\mu; p_i)$ up to pure gauge transformations. The symplectic 2-form is the Ω_{LW} confined to this manifold.
- The tangent space of the solution phase space is spanned (up to infinitesimal pure gauges) by "parametric variations":

$$\hat{\delta}\Phi \equiv \frac{\partial \hat{\Phi}}{\partial p_i}\delta p_i$$

● K. Hajian, A. Seraj and M. M. Sheikh-Jabbari, JHEP **1410**, 111 (2014), [arXiv:1407.1992].

Kerr-Newman solution phase space

- **Theory:** Einstein-Maxwell $\mathcal{L} = \frac{1}{16\pi G}(R - F^2)$
- **Dynamical fields $\hat{\Phi}$:** metric $g_{\mu\nu}$ and gauge field A_μ
- **Manifold $\hat{\mathcal{M}}$:**

$$\mathrm{d}s^2 = -(1-f)\mathrm{d}t^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 - 2fa\sin^2\theta\,\mathrm{d}t\mathrm{d}\psi + \left(r^2 + a^2 + fa^2\sin^2\theta\right)\sin^2\theta\,\mathrm{d}\psi^2\,,$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta\,, \qquad \Delta \equiv r^2 - 2Gmr + a^2 + q^2\,, \qquad f \equiv \frac{2Gmr - q^2}{\rho^2}\,,$$

....

$$A = \frac{q\,r}{\rho^2}\mathrm{d}t - \frac{q\,ra\sin^2\theta}{\rho^2}\mathrm{d}\psi\,, \qquad A \to A + \mathrm{d}\lambda$$

- **Parameters:** $p_i = \{m, a, q\}$
- **Parametric variations:**

$$\hat{\delta}g_{\mu\nu} = \frac{\partial\hat{g}_{\mu\nu}}{\partial m}\delta m + \frac{\partial\hat{g}_{\mu\nu}}{\partial a}\delta a + \frac{\partial\hat{g}_{\mu\nu}}{\partial q}\delta q\,, \qquad \hat{\delta}A_\mu = \frac{\partial\hat{A}_\mu}{\partial m}\delta m + \frac{\partial\hat{A}_\mu}{\partial a}\delta a + \frac{\partial\hat{A}_\mu}{\partial q}\delta q$$

3

Conserved charges associated with exact symmetries

Symplectic symmetry generator

Definition: A generator $\epsilon = \{\xi, \lambda\}$ is called symplectic symmetry generator if

(1) $\boldsymbol{\omega}(\delta\Phi, \delta_\epsilon\Phi, \Phi) \approx 0$ for all Φ and $\delta\Phi$ in the phase space and its tangent space,

(2) δH_ϵ be finite and integrable.

- Being a symplectic symmetry generator, conservation of δH_ϵ is guaranteed. It is because both of the equations are satisfied:

$$\mathrm{d}\boldsymbol{\omega}(\delta\Phi, \delta_\epsilon\Phi, \Phi) \approx 0\,, \qquad \boldsymbol{\omega}(\delta\Phi, \delta_\epsilon\Phi, \Phi)\Big|_{\partial\Sigma} \approx 0\,.$$

- Being a symplectic symmetry generator, the δH_ϵ is independent of chosen codimension-2 surface of integration:

$$\oint_{\mathcal{S}_2} \boldsymbol{k}_\epsilon(\delta\Phi, \Phi) - \oint_{\mathcal{S}_1} \boldsymbol{k}_\epsilon(\delta\Phi, \Phi) = \int_\Sigma \boldsymbol{\omega}(\delta\Phi, \delta_\xi\Phi, \Phi) = 0$$

Exact symmetry generators

Non-exact and exact symplectic symmetry generators

Symplectic symmetry generators are composed of two sets:

1. **non-exact symmetries:** A symplectic symmetry generator $\chi = \{\xi, \lambda\}$ is called non-exact if $\delta_\chi \Phi \neq 0$ at least on one point of the phase space.
2. **exact symmetries:** A symplectic symmetry generator $\eta = \{\zeta, \lambda\}$ is called exact if $\delta_\eta \Phi = 0$ all over the phase space.

- Non-exact symplectic symmetries have been used to build "statistical phase space".
- Exact symplectic symmetries are in our main focus in the "solution phase space".

No ambiguity

- Conserved charges associated with the exact symmetries are unambiguous:

$$\boldsymbol{\omega}(\delta\Phi, \delta_\eta\Phi, \Phi) \rightarrow \boldsymbol{\omega}(\delta\Phi, \delta_\eta\Phi, \Phi) + \mathrm{d}\big(\delta_\eta \mathbf{Y}(\delta\Phi, \Phi) - \cancelto{0}{\delta \mathbf{Y}(\delta_\eta\Phi, \Phi)}\big)$$

In brief:
Covariant phase space method + solution phase space $\hat{\Phi}(x^\mu, p_i)$ and parametric variations $\hat{\delta}\Phi$ + focusing on exact symmetry generators $\eta = \{\zeta, \lambda\}$:

Conserved charge associated with η and its integrability

Conserved charge associated with the exact symmetry $\eta = \{\zeta, \lambda\}$:

$$\hat{\delta} H_\eta = \oint_{\mathcal{S}} \boldsymbol{k}_\eta(\hat{\delta}\Phi, \hat{\Phi}) \,.$$

Integrability condition:

$$\oint_{\mathcal{S}} \Big(\zeta \cdot \hat{\boldsymbol{\omega}}(\hat{\delta}_1\Phi, \hat{\delta}_2\Phi, \hat{\Phi}) + \boldsymbol{k}_{\hat{\delta}_1\eta}(\hat{\delta}_2\Phi, \Phi) - \boldsymbol{k}_{\hat{\delta}_2\eta}(\hat{\delta}_1\Phi, \hat{\Phi}) \Big) = 0, \quad \forall \hat{\delta}_{1,2}\Phi \text{ and } \forall \hat{\Phi}.$$

If integrable, then

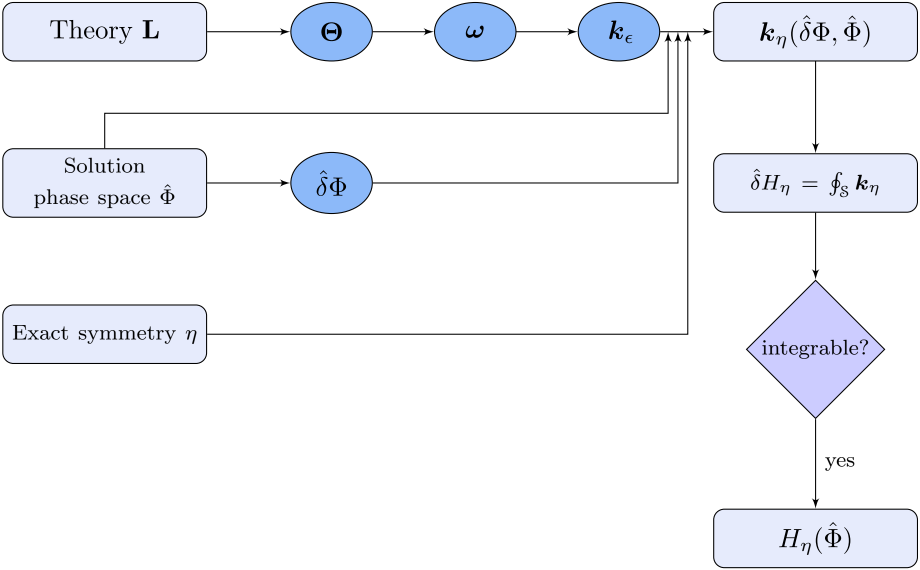
$$H_\eta[\hat{\Phi}] = \int_{\bar{p}}^{p} \hat{\delta} H_\eta + H_\eta[\bar{\Phi}] \,,$$

The $H_\eta[\bar{\Phi}]$ is the reference point (*i.e.* constant of integration).

Properties in brief:

- covariant phase space method $\Rightarrow$ covariant variations of charges,
- solution phase space $\Rightarrow$ calculability,
- exact symmetries $\Rightarrow$ conservation of charges,
- exact symmetries $\Rightarrow$ independence of the chosen codimension-2 surface of integration $\mathcal{S}$,
- exact symmetries $\Rightarrow$ removing ambiguity in the charge.

Overall view



4 Application: Kerr-Newman-(A)dS charges and first law(s)

- **Theory:** $\mathcal{L} = \frac{1}{16\pi G}(R - F^2 - 2\Lambda)$
- $\boldsymbol{k_\epsilon(\delta\Phi, \Phi)}$ **:** For the theory under consideration, and for diffeomorphism+gauge transformation $\epsilon = \{\xi, \lambda\}$

$$\boldsymbol{k}_\epsilon(\delta\Phi, \Phi) = \frac{\sqrt{-g}}{2!\,2!}\,\epsilon_{\mu\nu\sigma\rho}\,(k_\epsilon^{\mathrm{EH}\,\mu\nu} + k_\epsilon^{\mathrm{M}\,\mu\nu})\;\mathrm{d}x^\sigma \wedge \mathrm{d}x^\rho$$

in which

$$k_\xi^{\mathrm{EH}\,\mu\nu} = \frac{1}{16\pi G}\Big(\Big[\xi^\nu\nabla^\mu h - \xi^\nu\nabla_\tau h^{\mu\tau} + \xi_\tau\nabla^\nu h^{\mu\tau} + \frac{1}{2}h\nabla^\nu\xi^\mu - h^{\tau\nu}\nabla_\tau\xi^\mu\Big] - [\mu \leftrightarrow \nu]\Big)\,,$$

$$k_\epsilon^{\mathrm{M}\,\mu\nu} = \frac{1}{8\pi G}\Big(\Big[(\frac{-h}{2}F^{\mu\nu} + 2F^{\mu\rho}h_\rho{}^\nu - \delta F^{\mu\nu})(\xi^\sigma A_\sigma + \lambda) - F^{\mu\nu}\xi^\rho\delta A_\rho - 2F^{\rho\mu}\xi^\nu\delta A_\rho\Big] - [\mu \leftrightarrow \nu]\Big)$$

where $h^{\mu\nu} \equiv g^{\mu\sigma}g^{\nu\tau}\delta g_{\sigma\tau}$ and $h \equiv h^\mu{}_\mu$.

Application: Kerr-Newman-(A)dS black holes

- **Solution phase space $\hat{\mathcal{M}}$:**

$$ds^2 = -\Delta_\theta\Big(\frac{1-\frac{\Lambda r^2}{3}}{\Xi} - \Delta_\theta f\Big)dt^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 - 2\Delta_\theta f a \sin^2\theta\, dt d\varphi + \left(\frac{r^2+a^2}{\Xi} + f a^2 \sin^2\theta\right)\sin^2\theta\, d\varphi^2\,,$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta\,, \qquad \Delta_r \equiv (r^2+a^2)(1-\frac{\Lambda r^2}{3}) - 2Gmr + q^2\,,$$

$$\Delta_\theta \equiv 1 + \frac{\Lambda a^2}{3}\cos^2\theta\,, \qquad \Xi \equiv 1 + \frac{\Lambda a^2}{3}\,, \qquad f \equiv \frac{2Gmr}{\rho^2\Xi^2}\,,$$

$$\hat{A}_\mu dx^\mu = \frac{qr}{\rho^2\Xi}(\Delta_\theta dt - a\sin^2\theta\, d\varphi)\,.$$

- **Parameters:** $p_i = \{m, a, q\}$
- **Parametric variations:**

$$\hat{\delta} g_{\mu\nu} = \frac{\partial \hat{g}_{\mu\nu}}{\partial m}\delta m + \frac{\partial \hat{g}_{\mu\nu}}{\partial a}\delta a + \frac{\partial \hat{g}_{\mu\nu}}{\partial q}\delta q\,, \qquad \hat{\delta} A_\mu = \frac{\partial \hat{A}_\mu}{\partial m}\delta m + \frac{\partial \hat{A}_\mu}{\partial a}\delta a + \frac{\partial \hat{A}_\mu}{\partial q}\delta q$$

Application: Kerr-Newman-(A)dS black holes

- *Mass:* $\eta = \{\partial_t, 0\}$

$$\hat{\delta} M = \frac{\partial\left(\frac{m}{\Xi^2}\right)}{\partial m}\delta m + \frac{\partial\left(\frac{m}{\Xi^2}\right)}{\partial a}\delta a + \frac{\partial\left(\frac{m}{\Xi^2}\right)}{\partial q}\delta q \qquad \Rightarrow \qquad M = \frac{m}{\Xi^2}\,,$$

The reference points have been chosen such that pure (A)dS spacetime would have vanishing mass.

- *Angular momentum:* $\eta = \{\partial_\varphi, 0\}$

$$\hat{\delta} J = \frac{\partial\left(\frac{ma}{\Xi^2}\right)}{\partial m}\delta m + \frac{\partial\left(\frac{ma}{\Xi^2}\right)}{\partial a}\delta a + \frac{\partial\left(\frac{ma}{\Xi^2}\right)}{\partial q}\delta q \qquad \Rightarrow \qquad J = \frac{ma}{\Xi^2}\,.$$

The reference points have been chosen such that pure (A)dS spacetime would have vanishing angular momentum.

- *Electric charge:* $\eta = \{0, 1\}$

$$\hat{\delta} Q = \frac{\partial\left(\frac{q}{\Xi}\right)}{\partial m}\delta m + \frac{\partial\left(\frac{q}{\Xi}\right)}{\partial a}\delta a + \frac{\partial\left(\frac{q}{\Xi}\right)}{\partial q}\delta q \qquad \Rightarrow \qquad Q = \frac{q}{\Xi}\,.$$

The reference points have been chosen such that pure (A)dS spacetime would have vanishing electric charge.

Application: Kerr-Newman-(A)dS black holes

Choosing $r_{\rm H}$ to be any one of the horizons present in the geometry, then, surface gravity, angular velocity and electric potential associated to that horizon are:

$$\kappa_{\rm H} = \frac{r_{\rm H}(1 - \frac{\Lambda a^2}{3} - \Lambda r_{\rm H}^2 - \frac{a^2+q^2}{r_{\rm H}^2})}{2(r_{\rm H}^2 + a^2)}\,, \qquad \Omega_{\rm H} = \frac{a(1 - \frac{r_{\rm H}^2}{l^2})}{r_{\rm H}^2 + a^2}\,, \qquad \Phi_{\rm H} = \frac{q r_{\rm H}}{r_{\rm H}^2 + a^2}.$$

- *Entropies:* $\eta_{\rm H} = \{\zeta_{\rm H}, -\frac{2\pi\Phi_{\rm H}}{\kappa_{\rm H}}\}$ in which $\zeta_{\rm H} = \frac{2\pi}{\kappa_{\rm H}}(\partial_t + \Omega_{\rm H}\partial_\varphi)$

$$\hat{\delta} S_{\rm H} = \frac{\partial\left(\frac{\pi(r_{\rm H}^2+a^2)}{G\,\Xi}\right)}{\partial m}\delta m + \frac{\partial\left(\frac{\pi(r_{\rm H}^2+a^2)}{G\,\Xi}\right)}{\partial a}\delta a + \frac{\partial\left(\frac{\pi(r_{\rm H}^2+a^2)}{G\,\Xi}\right)}{\partial q}\delta q\,,$$

$$\Rightarrow \qquad S_{\rm H} = \frac{\pi(r_{\rm H}^2 + a^2)}{G\,\Xi}\,.$$

Reference points:

- Event horizons: vanishing entropy as reference point on pure (A)dS.
- Cosmological horizons: reference point on pure dS

$$H_{\eta_{\rm H}}[{\rm dS}_4] = \frac{\pi l^2}{G}$$

Application: Kerr-Newman-(A)dS black holes

$$\eta_{\text{H}} = \{\zeta_{\text{H}}, -\frac{2\pi\Phi_{\text{H}}}{\kappa_{\text{H}}}\} = \frac{2\pi}{\kappa_{\text{H}}}\{\partial_t, 0\} + \frac{2\pi\Omega_{\text{H}}}{\kappa_{\text{H}}}\{\partial_\varphi, 0\} - \frac{2\pi\Phi_{\text{H}}}{\kappa_{\text{H}}}\{0, 1\}\,,$$

- *First law(s):* linearity of δH_η in η, for each one of the horizons, results to

$$\delta S_{\text{H}} = \frac{2\pi}{\kappa_{\text{H}}}\delta M - \frac{2\pi}{\kappa_{\text{H}}}\Omega_{\text{H}}\delta J - \frac{2\pi}{\kappa_{\text{H}}}\Phi_{\text{H}}\delta Q$$

which by Hawking temperature(s) $T_{\text{H}} = \frac{\kappa_{\text{H}}}{2\pi}$ yields the first law(s)

$$\delta M = T_{\text{H}}\delta S_{\text{H}} + \Omega_{\text{H}}\delta J + \Phi_{\text{H}}\delta Q\,.$$

- Notice that although $T_{\mu\nu} \neq 0$, one does not need integrate it over the volume, to prove the first laws,
- By the integrability condition, one can rule out other Killing vectors as candidates for the mass etc,
- Charges are automatically finite.
- Entropy are found similar to other conserved charges, over almost arbitrary codimension-2 surface $\mathcal{S}$.
- thermodynamics of Kerr-AdS, Kerr and Kerr-dS are unified.

Thanks for your attention